Analiza 2
Rešitve 1. sklopa nalog

Nedoločeni integral

(1) Izračunaj integrale s pomočjo tabele elementarnih integralov:

(a) $\int \left(\sqrt{x}+1\right)\left(x-\sqrt{x}+1\right)\,dx$,

(b) $\int \sqrt{x\sqrt{x\sqrt{x}}}\,dx$,

(c) $\int 5^x 3^{-x}\,dx$,

(d) $\int \frac{x^2-1}{x^2+1}\,dx$,

(e) $\int \operatorname{tg}^2 x\,dx$.

Rešitev:

(a) $$\int \left(\sqrt{x}+1\right)\left(x-\sqrt{x}+1\right)\,dx = \int \left(x\sqrt{x}-x+\sqrt{x}+x-\sqrt{x}+1\right)\,dx = \underline{\underline{\frac{2}{5}x^{\frac{5}{2}}+x+C}}.$$

(b) $$\int \sqrt{x\sqrt{x\sqrt{x}}}\,dx = \int x^{\frac{7}{8}}\,dx = \underline{\underline{\frac{8}{15}x^{\frac{15}{8}}+C}}.$$

(c) $$\int 5^x 3^{-x}\,dx = \int \left(\frac{5}{3}\right)^x dx = \underline{\underline{\frac{1}{\ln\frac{5}{3}}\left(\frac{5}{3}\right)^x + C}}.$$

(d) $$\int \frac{x^2-1}{x^2+1}\,dx = \int \frac{x^2+1-2}{x^2+1}\,dx = \int \left(1-\frac{2}{x^2+1}\right)dx = \underline{\underline{x-2\operatorname{arc\,tg} x+C}}.$$

(e) $$\int \operatorname{tg}^2 x\,dx = \int \left(\frac{1}{\cos^2 x}-1\right)dx = \underline{\underline{\operatorname{tg} x - x + C}}.$$

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(2) Izračunaj integrale s pomočjo substitucije:

(a) $\displaystyle\int (5-2x)^9\,dx$,

(b) $\displaystyle\int \frac{1}{x^2+9}\,dx \quad \left(\text{splošno } \int \frac{1}{a^2x^2+b^2}\,dx\right)$,

(c) $\displaystyle\int \frac{1}{1+\cos x}\,dx$,

(d) $\displaystyle\int \frac{\cos x}{2+\sin x}\,dx$,

(e) $\displaystyle\int \frac{2x+2}{x^2+1}\,dx$,

(f) $\displaystyle\int \frac{1}{\sqrt{1+e^{2x}}}\,dx$.

Rešitev:

(a) $\displaystyle\int (5-2x)^9\,dx$:

Vzemimo novo spremenljivko $t = 5-2x$. Sledi $dt = -2dx$ in

$$\int (5-2x)^9\,dx = -\frac{1}{2}\int t^9\,dt = -\frac{1}{20}t^{10} + C = \underline{\underline{-\frac{1}{20}(5-2x)^{10} + C.}}$$

(b) $\displaystyle\int \frac{1}{x^2+9}\,dx \quad \left(\text{splošno } \int \frac{1}{a^2x^2+b^2}\,dx\right)$:

Vzemimo novo spremenljivko $t = \frac{x}{3}$. Potem je $dt = \frac{dx}{3}$ in

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx = \frac{1}{3}\int \frac{1}{t^2+1}\,dt = \underline{\underline{\frac{1}{3}\operatorname{arc\,tg}\frac{x}{3} + C.}}$$

V splošnem primeru vzemimo $t = \frac{ax}{b}$, kar nam da $dt = \frac{a\,dx}{b}$. Sledi

$$\int \frac{1}{a^2x^2+b^2}\,dx = \frac{1}{b^2}\int \frac{1}{\left(\frac{ax}{b}\right)^2+1}\,dx = \frac{b}{b^2a}\int \frac{1}{t^2+1}\,dt = \underline{\underline{\frac{1}{ab}\operatorname{arc\,tg}\frac{ax}{b} + C.}}$$

(c) $\displaystyle\int \frac{1}{1+\cos x}\,dx$:

Za izračun zadnjega integrala na desni vzemimo novo spremenljivko $t = \sin x$.

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}\,dx = \int \frac{1-\cos x}{\sin^2 x}\,dx = \underline{\underline{-\operatorname{ctg} x + \frac{1}{\sin x} + C.}}$$

(d) $\int \frac{\cos x}{2+\sin x}\, dx$:

Poskusimo z novo spremenljivko $t = 2+\sin x$. Potem je $dt = \cos x\, dx$ in

$$\int \frac{\cos x}{2+\sin x}\, dx = \int \frac{dt}{t} = \ln|t| + C = \underline{\underline{\ln|2+\sin x| + C}}.$$

Opomba: Včasih integriramo funkcije oblike $f(x) = \frac{g'(x)}{g(x)}$, kjer je g neka funkcija. V takih primerih uvedemo novo spremenljivko $u = g(x)$ (sledi $du = g'(x)dx$), da dobimo

$$\int f(x)\, dx = \int \frac{g'(x)}{g(x)}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|g(x)| + C.$$

(e) $\int \frac{2x+2}{x^2+1}\, dx$:

Uvedimo novo spremenljivko $t = x^2+1$. Sledi $dt = 2x\, dx$ in

$$\int \frac{2x+2}{x^2+1}\, dx = \int \left(\frac{2x}{x^2+1} + \frac{2}{x^2+1}\right) dx = \underline{\underline{\ln\left(x^2+1\right) + 2\operatorname{arc\,tg} x + C}}.$$

(f) $\int \frac{1}{\sqrt{1+e^{2x}}}\, dx$:

Definirajmo $t = \sqrt{1+e^{2x}}$. Potem je $dt = \frac{e^{2x}dx}{\sqrt{1+e^{2x}}}$ oziroma $\frac{dt}{t^2-1} = \frac{dx}{\sqrt{1+e^{2x}}}$. Sledi

$$\begin{aligned}\int \frac{1}{\sqrt{1+e^{2x}}}\, dx &= \int \frac{1}{t^2-1}\, dt = \frac{1}{2}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt,\\ &= \underline{\underline{\frac{1}{2}\ln\left(\sqrt{1+e^{2x}}-1\right) - \frac{1}{2}\ln\left(\sqrt{1+e^{2x}}+1\right) + C}}.\end{aligned}$$

Opomba: Alternativno bi lahko uporabili, da velja

$$\int \frac{1}{1-t^2}\, dt = \begin{cases} \operatorname{arcth} t + C & ;\ |t| > 1, \\ \operatorname{arth} t + C & ;\ |t| < 1, \end{cases}$$

kar nam da

$$\int \frac{1}{\sqrt{1+e^{2x}}}\, dx = -\operatorname{arcth}\left(\sqrt{1+e^{2x}}\right) + C.$$

Funkciji arcth in arth sta area kotangens in area tangens. $\square$

(3) Izračunaj integrale s pomočjo integracije po delih:

(a) $\displaystyle\int \ln x\,dx$,

(b) $\displaystyle\int \operatorname{arc\,tg} x\,dx$,

(c) $\displaystyle\int x^2 e^{-x}\,dx$,

(d) $\displaystyle\int e^{ax}\sin bx\,dx$,

(e) $\displaystyle\int x\sin x\cos x\,dx$.

Rešitev: Pri integraciji po delih si pomagamo s formulo

$$\int u\,dv = uv - \int v\,du.$$

Ponavadi se pri izbiri u in dv ravnamo po načelu:

$\cdot\, u \ldots$ funkcija, ki se pri odvajanju poenostavi,
$\cdot\, dv \ldots$ izraz, ki ga znamo integrirati.

(a) $\displaystyle\int \ln x\,dx$:

Vzemimo $u = \ln x$ in $dv = dx$. Sledi $du = \frac{dx}{x}$ in $v = x$.Tako dobimo

$$\int \ln x\,dx = x\ln x - \int x\,\frac{dx}{x} = \underline{\underline{x\ln x - x + C}}.$$

(b) $\displaystyle\int \operatorname{arc\,tg} x\,dx$:

Funkcijo $\operatorname{arc\,tg} x$ bomo odvajali, izraz dx pa integrirali. V dobljenem integralu bomo nato uvedli novo spremenljivko $t = x^2+1$. Sledi

$$\int \operatorname{arc\,tg} x\,dx = x\operatorname{arc\,tg} x - \int \frac{x}{x^2+1}\,dx = x\operatorname{arc\,tg} x - \frac{1}{2}\int \frac{dt}{t},$$
$$= \underline{\underline{x\operatorname{arc\,tg} x - \frac{1}{2}\ln|x^2+1| + C}}.$$

(c) $\displaystyle\int x^2 e^{-x}\,dx$:

Pri tem integralu bomo funkcijo e^{-x} dvakrat integrirali, monoma pa dvakrat odvajali.

$$\int x^2e^{-x}\,dx = -x^2e^{-x} - \int 2x(-e^{-x})\,dx = -x^2e^{-x} + 2\left(-xe^{-x} - \int(-e^{-x})\,dx\right),$$
$$= \underline{\underline{(-x^2-2x-2)e^{-x} + C}}.$$

Opomba: Na podoben način lahko izračunamo integrale oblike

$$\int p(x)e^{kx}\,dx,$$

kjer je p poljuben polinom in k poljubno realno število.

(d) $\int e^{ax}\sin bx\,dx$:

Pri tem integralu bomo eksponentno funkcijo dvakrat integrirali, trigonometrični funkciji pa dvakrat odvajali.

$$\begin{aligned}
\int e^{ax}\sin bx\,dx &= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx,\\
&= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\left(\frac{1}{a}e^{ax}\cos bx - \frac{b}{a}\int e^{ax}(-\sin bx)\,dx\right),\\
&= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx.
\end{aligned}$$

Iz te implicitne oblike lahko izrazimo

$$\int e^{ax}\sin bx\,dx = \underline{\underline{\frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C.}}$$

Na podoben način lahko izračunamo tudi

$$\int e^{ax}\cos bx\,dx = \underline{\underline{\frac{e^{ax}(b\sin bx + a\cos bx)}{a^2+b^2} + C.}}$$

Opomba: Poleg integriranja realnih funkcij poznamo tudi integriranje kompleksnih funkcij. Poglejmo si, kako bi lahko na ta način izračunali zgornja dva integrala. Naj bo $\lambda = a+ib$ kompleksno število. Potem velja

$$e^{\lambda x} = e^{(a+ib)x} = e^{ax}(\cos bx + i\sin bx).$$

Če označimo

$$I_c = \int e^{ax}\cos bx\,dx \text{ in } I_s = \int e^{ax}\sin bx\,dx,$$

je torej

$$\int e^{\lambda x}\,dx = I_c + iI_s.$$

Po drugi strani pa je

$$
\begin{aligned}
\int e^{\lambda x}\,dx &= \frac{1}{\lambda}e^{\lambda x}+C,\\
&= \frac{1}{a+ib}e^{ax}(\cos bx + i\sin bx)+C,\\
&= \frac{e^{ax}}{a^2+b^2}(a-ib)(\cos bx+i\sin bx)+C,\\
&= \frac{e^{ax}}{a^2+b^2}\big((b\sin bx + a\cos bx)+i(a\sin bx - b\cos bx)\big)+C,\\
&= \frac{e^{ax}(b\sin bx+a\cos bx)}{a^2+b^2}+i\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}+C.
\end{aligned}
$$

Vidimo, da smo izračunali oba integrala hkrati, ne da bi uporabili metodo integriranja po delih. Cena, ki smo jo plačali, pa je, da operiramo s kompleksnimi namesto z realnimi funkcijami.

(e) $\int x\sin x\cos x\,dx$:

Pri tem integralu bomo funkcijo x odvajali, funkcijo $\sin 2x$ pa integrirali.

$$
\begin{aligned}
\int x\sin x\cos x\,dx = \frac{1}{2}\int x\sin 2x\,dx &= \frac{1}{2}\left(-\frac{1}{2}x\cos 2x+\frac{1}{2}\int \cos 2x\,dx\right),\\
&= \underline{\underline{-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x + C.}}
\end{aligned}
$$

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(4) Izračunaj integrale:

(a) $\int x^3 e^{x^2}\,dx$,

(b) $\int \arcsin x\,dx$,

(c) $\int \frac{\ln(\ln x)}{x}\,dx$.

Rešitev:

(a) $\int x^3 e^{x^2}\,dx$:

Vzemimo novo spremenljivko $t=x^2$ (sledi $dt = 2x\,dx$), nato pa še integriramo po delih

$$
\begin{aligned}
\int x^3e^{x^2}\,dx = \frac{1}{2}\int te^t\,dt &= \frac{1}{2}\left(te^t-\int e^t\,dt\right),\\
&= \frac{1}{2}\left(te^t-e^t\right)+C = \underline{\underline{\frac{1}{2}e^{x^2}\left(x^2-1\right)+C.}}
\end{aligned}
$$

(b) $\displaystyle\int \operatorname{arc}\sin x \, dx$:

Najprej integriramo po delih $u = \arcsin x$, $dv = dx$, nato pa uvedimo $t = 1 - x^2$.

$$\begin{aligned} \int \arcsin x \, dx &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx = x \arcsin x + \frac{1}{2} \int t^{-1/2} \, dt, \\ &= x \arcsin x + \sqrt{t} + C = \underline{\underline{x \arcsin x + \sqrt{1-x^2} + C}}. \end{aligned}$$

(c) $\displaystyle\int \frac{\ln(\ln x)}{x} \, dx$:

Definirajmo $t = \ln x$, kar nam da $dt = \frac{dx}{x}$, nato pa integriramo po delih

$$\int \frac{\ln(\ln x)}{x} \, dx = \int \ln t \, dt = t \ln t - \int dt = t(\ln t - 1) + C = \underline{\underline{\ln x \, (\ln(\ln x) - 1) + C}}.$$

$\square$