Dynkin (λ-) and π-systems

Matija Vidmar

March 10, 2015

Definition 1. Let Ω be a set, $\mathcal{D} \subset \mathcal{P}(\Omega)$ a collection of its subsets. Then $\mathcal{D}$ is called a Dynkin system, or a λ-system, on Ω, if:

1. $\Omega \in \mathcal{D}$;
2. $\{A, B\} \subset \mathcal{D}$ and $A \subset B$, implies $B \backslash A \in \mathcal{D}$;
3. $(A_i)_{i \geq 1} \subset \mathcal{D}$, and $A_i \subset A_{i+1}$ for all $i \geq 1$, implies $\cup_{i \geq 1} A_i \in \mathcal{D}$.

Remark 2. Let Ω be a set, $\mathcal{D} \subset \mathcal{P}(\Omega)$ a collection of its subsets. Then $\mathcal{D}$ is a Dynkin system on Ω, if and only if:

1. $\Omega \in \mathcal{D}$;
2. $A \in \mathcal{D}$ implies $\Omega \backslash A \in \mathcal{D}$;
3. $(A_i)_{i \geq 1} \subset \mathcal{D}$, and $A_i \cap A_j = \emptyset$ for all $i \neq j$, implies $\cup_{i \geq 1} A_i \in \mathcal{D}$.

[If $\mathcal{D}$ is a Dynkin system, then for $\{A, B\} \subset \mathcal{D}$ with $A \cap B = \emptyset$, $A \cup B = \Omega \backslash ((\Omega \backslash A) \backslash B) \in \mathcal{D}$. Conversely, if the above hold, then for $\{A, B\} \subset \mathcal{D}$ with $A \subset B$, $B \backslash A = \Omega \backslash (A \cup (\Omega \backslash B)) \in \mathcal{D}$. The rest is trivial.]

Remark 3. Given a set Ω and a collection $\mathcal{L} \subset \mathcal{P}(\Omega)$ of its subsets, there is a smallest (with respect to inclusion) λ-system on Ω containing $\mathcal{L}$. It is the intersection of all the λ-systems on Ω containing $\mathcal{L}$ and will be denoted $\lambda_\Omega(\mathcal{L})$. (Similarly $\sigma_\Omega(\mathcal{L})$ will denote the smallest (with respect to inclusion) σ-field on Ω containing $\mathcal{L}$.)

Definition 4. Let Ω be a set, $\mathcal{L} \subset \mathcal{P}(\Omega)$ a collection of its subsets. Then $\mathcal{L}$ is called a π-system on Ω, if $A \cap B \in \mathcal{L}$, whenever $\{A, B\} \subset \mathcal{L}$.

Remark 5. Let Ω be a set, $\mathcal{D} \subset \mathcal{P}(\Omega)$ a collection of its subsets, which is both a π-system and a λ-system on Ω. Then $\mathcal{D}$ is a σ-algebra on Ω.

Theorem 6. *Let $\mathcal{L}$ be a π-sytem on Ω. Then $\sigma_\Omega(\mathcal{L}) = \lambda_\Omega(\mathcal{L})$.*

Proof. Since every σ-algebra on Ω is a λ-system on Ω, the inclusion $\sigma_\Omega(\mathcal{L}) \supset \lambda_\Omega(\mathcal{L})$ is manifest. For the reverse inclusion, it will be sufficient (and, indeed, necessary) to check $\lambda_\Omega(\mathcal{L})$ is a σ-algebra. Then it will be sufficient to check $\lambda_\Omega(\mathcal{L})$ is a π-system. Define $\mathcal{U} := \{A \in \lambda_\Omega(\mathcal{L}) : A \cap B \in \lambda_\Omega(\mathcal{L}) \text{ for all } B \in \mathcal{L}\}$: $\mathcal{U}$ is a λ-system, containing $\mathcal{L}$, so $\lambda_\Omega(\mathcal{L}) \subset \mathcal{U}$. Now define $\mathcal{V} := \{A \in \lambda_\Omega(\mathcal{L}) : A \cap B \in \lambda_\Omega(\mathcal{L}) \text{ for all } B \in \lambda_\Omega(\mathcal{L})\}$. Again $\mathcal{V}$ is a λ-system, containing, by what we have just shown, $\mathcal{L}$. But then $\lambda_\Omega(\mathcal{L}) \subset \mathcal{V}$, and we conclude. □

Corollary 7 (π-λ theorem)**.** *Let $\mathcal{L}$ be a π-system, $\mathcal{D}$ a Dynkin system on Ω, $\mathcal{L} \subset \mathcal{D}$. Then $\sigma_\Omega(\mathcal{L}) \subset \mathcal{D}$.*

Proof. Immediate. □

Some probabilistic corollaries follow.

"It suffices to check densities on a π-system."

Corollary 8. *Let X be a random element on some probabilty space $(\Omega, \mathcal{F}, \mathsf{P})$, with values in a σ-finite measure space (D, Σ, μ) (so X is $\mathcal{F}/\Sigma$-measurable). Suppose $\mathcal{L}$ is π-system on D, such that $\sigma_D(\mathcal{L}) = \lambda_D(\mathcal{L}) = \Sigma$; $f : D \to [0, +\infty]$ is a $\Sigma/\mathcal{B}([0, +\infty])$-measurable mapping with $\int f d\mu = 1$; and the following holds for all $L \in \mathcal{L}$:*

$$\mathsf{P}(X \in L) = \int_L f d\mu.$$

Then f is a density for X on (D, Σ, μ) under P, i.e. $\mathsf{P} \circ X^{-1} \ll \mu$ and $f = \frac{d(\mathsf{P} \circ X^{-1})}{d\mu}$ μ-a.e.[1]

Proof. From the hypotheses, linearity of integration and monotone convergence, the collection $\{L \in \Sigma : \mathsf{P}(X \in L) = \int_L f d\mu\}$ is a λ-system on D, containing the π-system $\mathcal{L}$. We conclude at once via the π-λ theorem. □

"A finite (in particular, a probability) measure is determined by its total mass and the values it assumes on a π-system."

Corollary 9. *Let μ and ν be two finite measures on a measurable space (E, Σ), $\mathcal{L} \subset \Sigma$ a π-system with $\sigma_E(\mathcal{L}) = \Sigma$. Suppose $\mu|_{\mathcal{L} \cup \{E\}} = \nu|_{\mathcal{L} \cup \{E\}}$. Then $\mu = \nu$.*

Proof. From the hypotheses, continuity from below, and finite additivity of measures, the collection of sets $\{A \in \Sigma : \mu(A) = \nu(A)\}$ is a λ-system on E containing $\mathcal{L}$. Enough said. □

Definition 10. Given a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and a collection $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ of subsets of $\mathcal{F}$, we say $\mathcal{C}$ is an independency (under P), if for any (non-empty) finite $I \subset \Lambda$, and then for any choices of $C_i \in \mathcal{C}_i$, $i \in I$, we have:

$$\mathsf{P}(\cap_{i \in I} C_i) = \prod_{i \in I} \mathsf{P}(C_i).$$

Of course, we say subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{F}$ are independent (under P), if the family $(\mathcal{A}, \mathcal{B})$ consisiting of them alone, is an independency (under P). Further, given a random element Z with values in a measurable space (E, Σ), and a subset $\mathcal{B}$ of $\mathcal{F}$, we say Z is independent of $\mathcal{B}$, if $\sigma(Z) := \{Z^{-1}(S) : S \in \Sigma\}$ (a subset of $\mathcal{F}$, since Z is a random element) is independent of $\mathcal{B}$. And so on, and so forth.

Remark 11. A sub-collection of an independency is an independency. A collection is an independency if and only if every finite sub-collection thereof is so.

"Independence can be raised from π-systems to the σ-fields they generate."

Corollary 12. *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ an independency under P consisting of π-systems alone. Then $(\sigma_\Omega(\mathcal{C}_\lambda))_{\lambda \in \Lambda}$ also is an independency under P.*

Proof. It will suffice to show that if the finite collection of π-systems $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ ($n \geq 2$ an integer) on Ω is an independency, then $(\sigma_\Omega(\mathcal{A}_1), \mathcal{A}_2, \ldots, \mathcal{A}_n)$ is one also. (Then one can apply mathematical induction.) Consider then $\mathcal{L}$, the collection of all subsets L of Ω, such that for all $A_2 \in \mathcal{A}_2$, …, $A_n \in \mathcal{A}_n$: $\mathsf{P}(L \cap A_2 \cap \cdots \cap A_n) = \mathsf{P}(L)\mathsf{P}(A_2) \cdots \mathsf{P}(A_n)$. It is, from the hypothesis and by properties of probability measures, a λ-system, containing the π-system $\mathcal{A}_1$. Enough said. □

[1] $\frac{d(\mathsf{P} \circ X^{-1})}{d\mu}$ is the Radon-Nikdoym derivative with respect to μ of the law of X under P on the space (D, Σ) (so $\mathsf{P} \circ X^{-1}(L) = \mathsf{P}(X \in L)$ for $L \in \Sigma$).