‘Renewal’ in renewal processes

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The aim of this note is to make precise the notion that a renewal process renews itself at each of the renewal epochs.

Let $T = (T_i)_{i\in\mathbb{N}}$ be a sequence of identically distributed $[0,\infty)$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, with $\mathsf{P}(T_1 = 0) < 1$, adapted to the filtration $\mathbb{F} = (\mathcal{F}_i)_{i\in\mathbb{N}_0}$, and such that T_{i+1} is independent of $\mathcal{F}_i$ for all $i \in \mathbb{N}_0$. Let $N = (N_t)_{t\geq 0}$ be the corresponding renewal process:

$$N_t := \sum_{k\in\mathbb{N}} \mathbb{1}(S_k \leq t) \text{ for } t \in [0,\infty),$$

where

$$S_n := \sum_{i=1}^{n} T_i \text{ for } n \in \mathbb{N}_0,$$

are the renewal epochs. Recall that we may, and shall, possibly by discarding a P-negligible set, insist on each N_t, $t \in [0,\infty)$, being $\mathbb{N}_0$-valued – equivalently on $S_n \uparrow \infty$ as $n \to \infty$. Interpret $S_\infty = \infty$.

Proposition 1. *Let Γ be an $\mathbb{F}$-stopping time, $\mathsf{P}(\Gamma < \infty) > 0$. Then $N' := (N_{S_\Gamma + t} - \Gamma)_{t\geq 0}$ defined on $\{\Gamma < \infty\}$ is independent of $\mathcal{F}_\Gamma|_{\{\Gamma<\infty\}}$ under the conditional measure $\mathsf{P}(\cdot|\Gamma < \infty)$, and has the same law thereunder as does N under P (on the space ${\mathbb{N}_0}^{[0,\infty)}$).*

Remark 1. An important case is that of Γ deterministic and finite.

Remark 2. When $N_0 = 0$ and N has jumps of size 1 (equivalently, T has values in $(0,\infty)^{\mathbb{N}}$), we find $N_{S_\Gamma} = \Gamma$, so N' is nothing but the incremental process $\Delta_{S_\Gamma} N$ of N after S_Γ. In that case we also have $N_t^{S_\Gamma} = \sum_{k=1}^{\Gamma} \mathbb{1}(S_k \leq t) = \sum_{k=1}^{\infty} \mathbb{1}_{\{k\leq\Gamma\}} \mathbb{1}_{[0,t]} \circ S_k$ for $t \in [0,\infty)$, and so $\sigma(N^{S_\Gamma}) \subset \mathcal{F}_\Gamma$, hence $\sigma(N^{S_\Gamma})|_{\{\Gamma<\infty\}}$ is independent of N'.[1] Under further ‘reasonable conditions’ on $(\Omega, \mathcal{F})$ (unclear whether always[2]), and if $\mathbb{F}$ is the natural filtration of T, letting $\mathcal{F}^N = (\mathcal{F}_t^N)_{t\geq 0}$ be the natural filtration of N, $\sigma(N^{S_\Gamma}) = \mathcal{F}^N_{S_\Gamma}$, whence the above says that, on $\{S_\Gamma < \infty\}$, under $\mathsf{P}(\cdot|S_\Gamma < \infty)$, the past of N up to S_Γ is independent of the incremental process $\Delta_{S_\Gamma} N$, which in turn has the same law as the renewal process we started off with – implying the strong Markov property at the (random) renewal epoch S_Γ![3] It is not to say N is strong Markov, though; indeed it is (in general) not even a Markov process.

Proof. Note that $T : \Omega \to [0,\infty)^{\mathbb{N}}$ is measurable, where $[0,\infty)^{\mathbb{N}}$ is endowed with the product σ-field $\otimes_{\mathbb{N}} \mathcal{B}([0,\infty))$. Similarly $N : \Omega \to {\mathbb{N}_0}^{[0,\infty)}$ is measurable, where ${\mathbb{N}_0}^{[0,\infty)}$ again is endowed with the product σ-field $\otimes_{[0,\infty)} 2^{\mathbb{N}_0}$.

Define $T' := (T_{\Gamma+i})_{i\in\mathbb{N}}$ on $\{\Gamma < \infty\}$. Then $T' : \Omega \to [0,\infty)^{\mathbb{N}}$ is measurable, independent of $\mathcal{F}_\Gamma|_{\{\Gamma<\infty\}}$ under $\mathsf{P}(\cdot|\Gamma < \infty)$, and its law (under $\mathsf{P}(\cdot|\Gamma < \infty)$, on the space $[0,\infty)^{\mathbb{N}}$) conicides

[1] For a process X and a random time S, X^S is the stopped process, $X_t^S = X_{S\wedge t}$, $t \in [0,\infty)$.

[2] See [arXiv:1503.02375, Part II] for details.

[3] If $\mathbb{F}$ is the natural filtration of T then S_Γ is an $\mathcal{F}^N$-stopping time: for $t \in [0,\infty)$ we have $\{S_\Gamma \leq t\} = \cup_{k\in\mathbb{N}_0} \{S_k \leq t\} \cap \{\Gamma = k\}$, whilst for each $k \in \mathbb{N}_0$, $\{\Gamma = k\} \in \mathcal{F}_k = \mathcal{F}_k^T = \mathcal{F}_k^S = \sigma(S_1, \ldots, S_k) \subset \mathcal{F}^N_{S_k}$ owing to the fact that $(S_l)_{l\in\mathbb{N}_0}$ is a nondecreasing sequence of $\mathcal{F}^N$-stopping times.

with that of T (under P, on the same space), by the strong Markov property for sequences with independent values. Define furthermore the mapping $G : [0,\infty)^{\mathbb{N}} \to {\mathbb{N}_0}^{[0,\infty)}$, as follows:

$$G(t)(s) = \left(\sum_{k\in\mathbb{N}} \mathbb{1}_{[0,s]}\left(\sum_{l=1}^{k} t_l\right)\right) \cdot \mathbb{1}_{\mathbb{N}_0}\left(\sum_{k\in\mathbb{N}} \mathbb{1}_{[0,s]}\left(\sum_{l=1}^{k} t_l\right)\right), \quad s \in [0,\infty), \quad t \in [0,\infty)^{\mathbb{N}}$$

(adhering to the convention $\infty \cdot 0 = 0$). Then G is measurable and $N = G \circ T$. (Incidentally, we recover again that N is measurable.)

Next we see that for each $t \in [0,\infty)$: $N'_t = \sum_{k\in\mathbb{N}} \mathbb{1}(S_k \leq S_\Gamma + t) - \Gamma = \sum_{k=\Gamma+1}^{\infty} \mathbb{1}(S_k - S_\Gamma \leq t) = \sum_{k=\Gamma+1}^{\infty} \mathbb{1}(\sum_{l=\Gamma+1}^{k} T_l \leq t) = \sum_{k=1}^{\infty} \mathbb{1}(\sum_{l=1}^{k} T'_l \leq t)$, hence $N' = G \circ T'$. In particular, N' is measurable.

Furthermore, since $N = G \circ T$, $N' = G \circ T'$, G is measurable and, under the relevant measures, the laws of T and T' agree, so the laws of N and N' agree also.

Finally, N' is independent of $\mathcal{F}_\Gamma|_{\{\Gamma<\infty\}}$ since it is a measurable function of T'. $\square$