Krepka lastnost Markova za zaporedja z neodvisnimi stacionarnimi vrednostmi

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Trditev. Naj bo $X = (X_i)_{i\in\mathbb{N}}$ zaporedje slučajnih elementov z vrednostmi v merljivem prostoru $(E, \mathcal{E})$ na verjetnostnem prostoru $(\Omega, \mathcal{G}, \mathsf{P})$. Dana sta še filtracija $\mathcal{F} = (\mathcal{F}_i)_{i\in\mathbb{N}_0}$ v diskretnem času in njen čas ustavljanja τ. Denimo da je X_{i+1} neodvisna od $\mathcal{F}_i$, $\mathcal{F}_{i+1}$-merljiva, in enako porazdeljena kot X_1, za vsak $i \in \mathbb{N}_0$; ter da je $\mathsf{P}(\tau < \infty) > 0$. Potem je zaporedje $\square_\tau X := (X_{\tau+i})_{i\in\mathbb{N}}$ definirano na $\{\tau < \infty\}$, pod pogojno mero $\mathsf{P}(\cdot|\tau < \infty)$, enako porazdeljeno[1] kot X pod mero P, in neodvisno od $\mathcal{F}_\tau|_{\{\tau<\infty\}}$.
Opomba. Tipično imamo zaporedje X n.e.p. slučajnih spremenljivk (z vrednostmi v $\mathbb{R}$), in za $\mathcal{F}$ vzamemo naravno filtracijo X-a. Običajno je tudi $\tau < \infty$ (morda samo s.g.), pa nam potem ni treba pogojevati na $\{\tau < \infty\}$.

Dokaz. Let $n \in \mathbb{N}$, $\{A_1, \ldots, A_n\} \subset \mathcal{E}$, $A \in \mathcal{F}_\tau$ and $k \in \mathbb{N}_0$. Then, owing to X being $\mathcal{F}$-adapted and having stationary independent values relative to $\mathcal{F}$, τ being a stopping time thereof, we have that $\mathsf{P}(X_{\tau+1} \in A_1, \ldots, X_{\tau+n} \in A_n, A, \tau = k) = \mathsf{P}(X_{k+1} \in A_1, \ldots, X_{k+n} \in A_n, A, \tau = k) = \mathsf{P}(X_{k+n} \in A_n) \cdots \mathsf{P}(X_{k+1} \in A_1)\mathsf{P}(A, \tau = k) = \mathsf{P}(X_1 \in A_1) \cdots \mathsf{P}(X_n \in A_n)\mathsf{P}(A, \tau = k) = \mathsf{P}(X_1 \in A_1, \ldots, X_n \in A_n)\mathsf{P}(A, \tau = k)$. Now $\sum_{k\in\mathbb{N}_0}$ and divide by $\mathsf{P}(\tau < \infty)$. We conclude $\mathsf{P}(X_{\tau+1} \in A_1, \ldots, X_{\tau+n} \in A_n, A|\tau < \infty) = \mathsf{P}(X_1 \in A_1, \ldots, X_n \in A_n)\mathsf{P}(A|\tau < \infty)$. It follows, via a ("monotone class") π/λ-argument that for any $B \in \mathcal{E}^{\otimes\mathbb{N}}$, $\mathsf{P}(\square_\tau X \in B, A|\tau < \infty) = \mathsf{P}(X \in B)\mathsf{P}(A|\tau < \infty)$. Taking $A = \Omega$ we find that the law of $\square_\tau X$ under $\mathsf{P}(\cdot|\tau < \infty)$ is the same as the law of X under P. In particular $\mathsf{P}(\square_\tau X \in B, A'|\tau < \infty) = \mathsf{P}(\square_\tau X \in B|\tau < \infty)\mathsf{P}(A'|\tau < \infty)$ for any $A' \in \mathcal{F}_\tau|_{\{\tau<\infty\}}$ and $B \in \mathcal{E}^{\otimes\mathbb{N}}$, i.e. $\square_\tau X$ is independent of $\mathcal{F}_\tau|_{\{\tau<\infty\}}$ under $\mathsf{P}(\cdot|\tau < \infty)$. $\square$

[1] Na prostoru $(E^{\mathbb{N}}, \mathcal{E}^{\otimes\mathbb{N}})$.