Porazdelitev (med)prihodnih časov v Poissonovem procesu

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Naj bo N Poissonov proces s prihodnimi časi $(S_n)_{n\in\mathbb{N}}$, medprihodnimi časi $(T_n)_{n\in\mathbb{N}}$, in z zvezno funkcijo intenzitete ρ, ki ni identično enaka nič – ekvivalentno, $\mathsf{P}(S_n<\infty)>0$ za nek (potem vsak) $n\in\mathbb{N}$. Označimo $\Theta(t):=\int_0^t\rho(s)ds$ za $t\in[0,\infty]$. Spomnimo da je $\Theta(\infty)=\infty$, natanko tedaj ko je $\mathsf{P}(S_n<\infty)=1$ za nek (potem vsak) $n\in\mathbb{N}$. Končno naj bo $\Delta_n:=\{(s_1,\ldots,s_n)\in\mathbb{R}^n:0<s_1<\cdots<s_n\}$.

Trditev. Naj bo $n\in\mathbb{N}$. Pogojno na $\{S_n<\infty\}$ je $(S_1,\ldots,S_n)$ absolutno zvezen slučajen vektor z gostoto (v $(s_1,\ldots,s_n)\in\mathbb{R}^n$)

$$\mathbb{1}_{\Delta_n}(s_1,\ldots,s_n)\frac{\rho(s_1)\cdots\rho(s_n)e^{-\Theta(s_n)}}{1-\sum_{k=0}^{n-1}\frac{\Theta(\infty)^k}{k!}e^{-\Theta(\infty)}},$$

kjer je potrebno imenovalec razumeti v limitnem smislu ko je $\Theta(\infty)=\infty$ (torej je tedaj enak 1).

Opombe.

1. Iz formule za transformacije gostot absolutno zveznih slučajnih vektorjev sledi, da je pogojno na $\{S_n<\infty\}$, $(T_1,\ldots,T_n)$ absolutno zvezen slučajen vektor z gostoto (v $(t_1,\ldots,t_n)\in\mathbb{R}^n$)

$$\mathbb{1}_{(0,\infty)^n}(t_1,\ldots,t_n)\frac{\rho(t_1)\cdots\rho(t_1+\cdots+t_n)e^{-\Theta(t_1+\cdots+t_n)}}{1-\sum_{k=0}^{n-1}\frac{\Theta(\infty)^k}{k!}e^{-\Theta(\infty)}}.$$

   Absolutna vrednost determinante Jacobija relevantne bijektivne linearne preslikave

$$G(t_1,\ldots,t_n):=(t_1,\ldots,t_1+\cdots+t_n)$$

   (za katero je torej $G\circ(T_1,\ldots,T_n)=(S_1,\ldots,S_n)$ na $\{S_n<\infty\}$) je namreč v tem primeru identično enaka 1.

2. Kadar je $\Theta(\infty)=\infty$, lahko brez vpliva na porazdelitve privzamemo, da je $S_n<\infty$ za vsak $n\in\mathbb{N}$ z gotovostjo.

Dokaz. Let $0=b_0<a_1\leq b_1<a_2\leq b_2<\cdots<a_n\leq b_n$ be real numbers. Then

$$\{(S_1,\ldots,S_n)\in\prod_{i=1}^n(a_i,b_i]\}$$

$$=\left(\cap_{i=1}^{n-1}\{N_{a_i}-N_{b_{i-1}}=0\}\cap\{N_{b_i}-N_{a_i}=1\}\right)\cap\{N_{a_n}-N_{b_{n-1}}=0\}\cap\{N_{b_n}-N_{a_n}\geq 1\},$$

hence from the independence of the increments in a PP and their known distribution

$$\mathsf{P}\left((S_1,\ldots,S_n)\in\prod_{i=1}^n(a_i,b_i]\right)$$

$$=\left(\prod_{i=1}^{n-1}e^{-(\Theta(a_i)-\Theta(b_{i-1}))}\left(\Theta\left(b_i\right)-\Theta\left(a_i\right)\right)e^{-(\Theta(b_i)-\Theta(a_i))}\right)e^{-(\Theta(a_n)-\Theta(b_{n-1}))}\left(1-e^{-(\Theta(b_n)-\Theta(a_n))}\right).$$

Telescopic cancellation yields the first equality in

$$\mathsf{P}\left((S_1,\dots,S_n)\in\prod_{i=1}^n(a_i,b_i]\right)=\prod_{i=1}^{n-1}\left(\Theta(b_i)-\Theta(a_i)\right)\left(e^{-\Theta(a_n)}-e^{-\Theta(b_n)}\right)$$
$$=\int_{\prod_{i=1}^n(a_i,b_i]}\rho(s_1)\cdots\rho(s_n)e^{-\Theta(s_n)}ds_1\cdots ds_n,$$

the second equality being a consequence of the fundamental theorem of calculus.

Now, for $k\in\mathbb{N}$, $\mathbb{1}(N_M=k)\to\mathbb{1}(N_\infty=k)$ /though not monotonically/ as $M\to\infty$ over $\mathbb{N}$, so from dominated convergence $\mathsf{P}(N_\infty=k)=\lim_{M\to\infty}\mathsf{P}(N_M=k)$. Hence $\mathsf{P}(S_n<\infty)=\mathsf{P}(N_\infty\geq n)=1-\mathsf{P}(N_\infty\leq n-1)=1-\sum_{k=0}^{n-1}\frac{\Theta(\infty)^k}{k!}e^{-\Theta(\infty)}$. It follows that

$$\mathsf{P}((S_1,\dots,S_n)\in A|S_n<\infty)=\int_A\frac{\rho(s_1)\cdots\rho(s_n)e^{-\Theta(s_n)}}{1-\sum_{k=0}^{n-1}\frac{\Theta(\infty)^k}{k!}e^{-\Theta(\infty)}}ds_1\cdots ds_n$$

for all $A\in\{\prod_{i=1}^n(a_i,b_i]:0=b_0<a_1\leq b_1<a_2\leq b_2<\cdots<a_n\leq b_n\text{ real}\}=:\Pi$, the latter being a π-system generating $\mathcal{B}(\Delta_n)$: to see that it is a π-system take $\prod_{i=1}^n(a_i,b_i]\in\Pi$ and $\prod_{i=1}^n(a_i',b_i']\in\Pi$ – then their intersection has the form $\prod_{i=1}^n((a_i\vee a_i')\wedge(b_i\wedge b_i'),b_i\wedge b_i']\in\Pi$; to see Π generates the Borel σ-field on Δ_n note that every point from Δ_n has a fundamental system of neighborhoods from Π with rational endpoints.

Moreover, by symmetry, Tonelli's theorem, and since the set $\{x\in(0,\infty)^n:x_i=x_j\text{ for some }i,j\}$ has Lebesgue measure zero,

$$n!\int_{\Delta_n}\rho(s_1)\cdots\rho(s_n)ds_1\cdots ds_n$$
$$=\int_{(0,\infty)^n\backslash\{x\in(0,\infty)^n:x_i=x_j\text{ for some }i,j\}}\rho(s_1)\cdots\rho(s_n)ds_1\cdots ds_n$$
$$=\int_{(0,\infty)^n}\rho(s_1)\cdots\rho(s_n)ds_1\cdots ds_n=\Theta(\infty)^n.$$

It is hence verified inductively on n that

$$\int_{\Delta_n}\rho(s_1)\cdots\rho(s_n)e^{-\Theta(s_n)}ds_1\cdots ds_n=1-\sum_{k=0}^{n-1}\frac{\Theta(\infty)^k}{k!}e^{-\Theta(\infty)}.$$

Then we obtain by an application of the π/λ theorem that in fact

$$\mathsf{P}((S_1,\dots,S_n)\in A|S_n<\infty)=\int_A\frac{\rho(s_1)\cdots\rho(s_n)e^{-\Theta(s_n)}}{1-\sum_{k=0}^{n-1}\frac{\Theta(\infty)^k}{k!}e^{-\Theta(\infty)}}ds_1\cdots ds_n$$

for all $A\in\mathcal{B}(\Delta_n)$. Noting that $(S_1,\dots,S_n)\in\Delta_n$ on $\{S_n<\infty\}$ it follows that for $A\in\mathcal{B}(\mathbb{R}^n)$

$$\mathsf{P}((S_1,\dots,S_n)\in A|S_n<\infty)=\int_{A\cap\Delta_n}\frac{\rho(s_1)\cdots\rho(s_n)e^{-\Theta(s_n)}}{1-\sum_{k=0}^{n-1}\frac{\Theta(\infty)^k}{k!}e^{-\Theta(\infty)}}ds_1\cdots ds_n$$
$$=\int_A\mathbb{1}_{\Delta_n}(s_1,\dots,s_n)\frac{\rho(s_1)\cdots\rho(s_n)e^{-\Theta(s_n)}}{1-\sum_{k=0}^{n-1}\frac{\Theta(\infty)^k}{k!}e^{-\Theta(\infty)}}ds_1\cdots ds_n,$$

which is the desired result. $\square$