Order statistics

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Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $X = (X_1, \ldots, X_n)$ a random vector with values in $\mathbb{R}^n$, $n \in \mathbb{N}$. Notation: $[n] := \{1, \ldots, n\}$, $\Gamma_n := \{y \in \mathbb{R}^n : y_1 < \cdots < y_n\}$. Define the order statistics of X via

$$X_{(i)} := \wedge_{J \in \binom{[n]}{i}} \vee_{j \in J} X_i \text{ for } i \in [n].$$

Let $\tilde{X} := (X_{(1)}, \ldots, X_{(n)})$ for short.

Suppose now the X_i, $i \in [n]$, are iid, absolutely continuous, with density f. Then for real $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$, from the symmetry of the X_is, $\mathsf{P}(\tilde{X} \in \prod_{i=1}^n (a_i, b_i)) = \int_{\prod_{i=1}^n (a_i, b_i)} n! f(x_1) \cdots f(x_n) dx_1 \cdots dx_n$. Since $\mathsf{P}(\tilde{X} \notin \Gamma_n) \leq \sum_{i,j=1}^n \mathsf{P}(X_i = X_j) = 0$; since $\Pi := \{\prod_{i=1}^n (a_i, b_i) : a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n \text{ real}\}$ is a π-system generating $\mathcal{B}(\Gamma_n)$; and since from symmetry, from $\{x \in \mathbb{R}^n : x_i = x_j \text{ for some } i, j\}$ having Lebesgue measure zero, and from Tonelli's theorem, one has

$$\int_{\Gamma_n} n! f(x_1) \cdots f(x_n) dx_1 \cdots dx_n = \int_{\mathbb{R}^n \backslash \{y \in (0,\infty)^n : y_i = y_j \text{ for some } i,j\}} f(x_1) \cdots f(x_n) dx_1 \cdots dx_n = 1;$$

it follows by an application of Dynkin's lemma that in fact

$$\mathsf{P}(\tilde{X} \in A) = \int_A n! f(x_1) \cdots f(x_n) dx_1 \cdots dx_n$$

for all $A \in \mathcal{B}(\Gamma_n)$. Again since $\mathsf{P}(\tilde{X} \in \Gamma_n) = 1$ this implies $\tilde{X}$ is absolutely continuous with density (in $(x_1, \ldots, x_n) \in \mathbb{R}^n$)

$$n! f(x_1) \cdots f(x_n) \mathbb{1}_{\Gamma_n}(x_1, \ldots, x_n).$$