Rešitve 1. izpita iz NM za fizike (22. 6. 16)

(1) a) $AXB = C \iff X = A^{-1} C B^{-1}$

Ker so A, B in C nesingularne, je točka tudi X.

b) Pišimo $XB = Y \Rightarrow AY = C$.

Zapišimo A in C po stolpcih:

$A = [a_1, a_2, \dots, a_n]$, $C = [c_1, c_2, \dots, c_n]$

$Y = [y_1, y_2, \dots, y_n]$

(i) Izvedemo LU razcep matrike A s pivotiranjem: $PA = LU$ $(\mathcal{O}(n^3))$

(ii) $PAY = PC \Rightarrow (LU)Y = PC$

Pišemo $UY = Z = [z_1, z_2, \dots, z_n]$

$\Rightarrow Lz_i = Pc_i$ (n krat $\mathcal{O}(n^2)$)

$Uy_i = z_i$ (n krat $\mathcal{O}(n^2)$)

$i = 1, 2, \dots, n$

(iii) Vrnemo se na $XB = Y \Rightarrow B^T X^T = Y^T$ in rešimo podobno kot v točkah (i), (ii).

Skupna časovna zahtevnost je $O(n^3)$. ②

c) V konkretnem primeru dobimo (brez pivotiranja)

$$Y = \begin{bmatrix} 7 & 10 \\ 17 & 24 \end{bmatrix} \text{ in } X^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Torej je $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

2 a) $$g(x) = (1 - \log x)\frac{x}{x+1}$$

$$g(\alpha) = \alpha \iff (1-\log\alpha)\frac{\alpha}{\alpha+1} = \alpha$$

$$\iff (1-\log\alpha)\cancel{\alpha} = (\alpha+1)\cancel{\alpha} \quad / : \alpha \ (\alpha \neq 0)$$

$$1 - \log\alpha = \alpha + 1$$

$$\log\alpha + \alpha = 0 \quad \checkmark$$

b) $$g'(x) = -\frac{1}{\cancel{x}}\cdot\frac{\cancel{x}}{x+1} + (1-\log x)\left(\frac{\cancel{x}+1-\cancel{x}}{(x+1)^2}\right)$$

$$= -\frac{1}{x+1} + (1-\log x)\frac{1}{(x+1)^2}$$

$$= \frac{-(x+1)+(1-\log x)}{(x+1)^2} = \frac{-\log x - x}{(x+1)^2} = -\frac{\log x + x}{(x+1)^2}$$

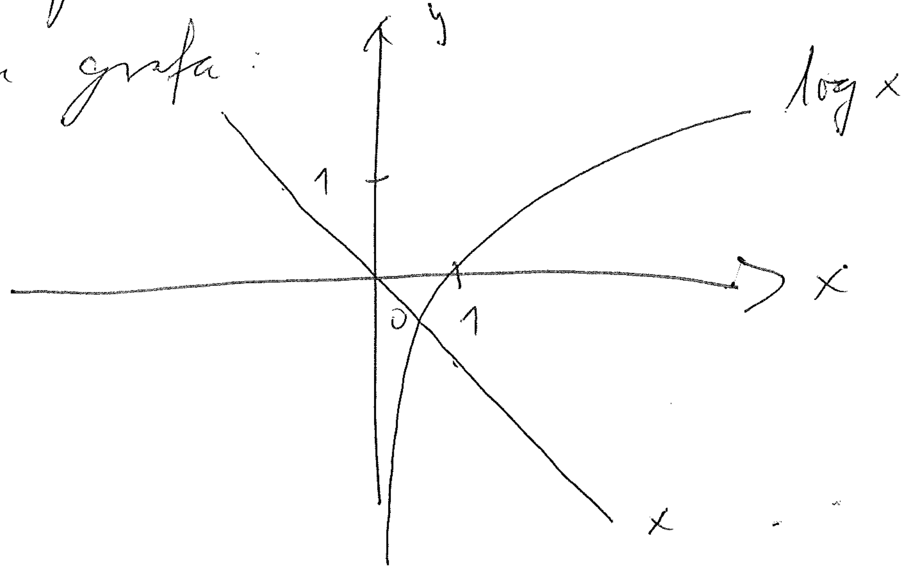
Če je $\alpha + \log \alpha = 0 \Rightarrow$

$$g'(\alpha) = -\frac{\log \alpha + \alpha}{(\alpha+1)^2} = 0 \Rightarrow$$

red konvergence je vsaj 2.

c) Rešujemo $x + \log x = 0 \Leftrightarrow \log x = -x$

Skica grafa:



Rešitev enačbe je med 0 in 1.

Za x_0 vzamemo $x_0 = 1/2$ in izvajamo

iteracijo $x_{r+1} = (1 - \log x_r)\left(\frac{x_r}{x_r + 1}\right)$

$x_0 = 0.5$, $x_1 = 0.5644$, $x_2 = 0.5671$, $x_3 = \underline{\underline{0.5671}}$

(3) Kot vemo, znamo poiskati Householderjevo zrcaljenje, recimo mu Q^T, ki prvi stolpec matrike A^T, torej $x = [1,1,1,1]^T$ preslika v $[k,0,0,0]^T$. To zrcaljenje je oblike

$$Q^T = I - \frac{2}{w^T w} w w^T, \quad \text{kjer}$$

je
$$w = \begin{bmatrix} x_1 + \operatorname{sign}(x_1)\, \|x\|_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Ker je $(Q^T A^T)^T = AQ$, bo Q torej preslikal prvo vrstico A v $[k,0,0,0]$.

Ker je $Q = Q^T$, je

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{\cancel{2}\ 1}{\cancel{12}\ 6} \begin{bmatrix} 9 & 3 & 3 & 3 \\ 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2, & -1/2, & -1/2, & -1/2 \\ -1/2, & 5/6, & -1/6, & -1/6 \\ -1/2, & -1/6, & 5/6, & -1/6 \\ -1/2, & -1/6, & -1/6, & 5/6 \end{bmatrix}.$$

④ $y'' = -y, \quad y(0) = 1, \quad y'(0) = 0$

$\cancel{y}\ y_1 = y, \ y_2 = y' \Rightarrow y_1' = y_2, \ y_2' = -y_1$

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Y' = \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix} \Rightarrow Y_{k+1} = Y_k + \frac{h}{2}\left(\begin{bmatrix} Y_k(2) \\ -Y_k(1) \end{bmatrix} + \begin{bmatrix} Y_{k+1}(2) \\ -Y_{k+1}(1) \end{bmatrix}\right)$$

Za $k=1$ dobimo sistem enačb (linearen):

$$Y_1(1) = Y_0(1) + \frac{1}{4}\left(Y_0(2) + Y_1(2)\right), \quad Y_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Y_1(2) = Y_0(2) + \frac{1}{4}\left(-Y_0(1) - Y_1(1)\right)$$

Neznanki sta $\alpha = Y_1(1)$, $\beta = Y_1(2) \Rightarrow$

$$\alpha = 1 + \frac{1}{4}(0 + \beta) \qquad 4\alpha - \beta = 4$$

$$\beta = 0 + \frac{1}{4}(-1 - \alpha) \Rightarrow \quad \alpha + 4\beta = -1 \ / \cdot 4$$

$$17\beta = -8$$

$$\beta = -\frac{8}{17} \doteq -0.47$$

Analitična rešitev je $\underline{\underline{y(x) = \sin x}}$.

$$\alpha = -1 - 4\beta = \frac{15}{17} \doteq 0.88$$