Rešitve 2. izpita iz NM – fiziki (6. 9. 2016)

(1) Matrika je očitno simetrična.

Razcep Choleskega L dobimo v treh korakih: $(A = LL^T$, L spodnja trikotna$)$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 13 & -9 \\ 3 & -9 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -2 & 9 & -3 \\ 3 & -3 & 5 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 3 & -1 & 2 \end{bmatrix} = L \quad \text{1T}$$

Ker se postopek izide, je matrika simetrična, pozitivno definitna.

Sedaj rešujemo $(LL^T)x = b$: T

(i) $Ly = b \sim \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 27 \end{bmatrix} \Rightarrow \begin{array}{l} y_1 = 6 \\ y_2 = 3 \\ y_3 = 6 \end{array}$

(ii) $L^T x = y \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} \Rightarrow \boxed{\begin{array}{l} x_3 = 3 \\ x_2 = 2 \\ x_1 = 1 \end{array}}$ T

② $f(x) = e^{-x^2} - x^3$

a)
(i) $x \leq 0 \Rightarrow e^{-x^2} - x^3 > 0 \Rightarrow$ ni rešitve na $(-\infty, 0]$ 5

(ii) $x > 0 \Rightarrow f'(x) = -2x e^{-x^2} - 3x^2 < 0$

$\Rightarrow f$ monotono pada $\Rightarrow$ <u>kvečjemu ena rešitev</u> 10

$f(0) = 1 > 0, \quad f(1) = \frac{1}{e} - 1 < 0$

$\Rightarrow$ ničla je na $[0, 1]$ in je edina

b) $$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} = x_r - \frac{e^{-x_r^2} - x_r^3}{-2x_r e^{-x_r^2} - 3x_r^2}$$ 5

$x_0 = 0.5, \quad x_1 = 0.9277, \quad x_2 = 0.8162,$

$x_3 = 0.80561, \quad x_4 = 0.80551, \quad x_5 = \underline{0.80551}$ 5

③ $x_0 = 0, \quad x_1 = 1, \quad x_2 = 2$

$$|f(x) - p_5(x)| = \frac{1}{6!}\,|f^{(6)}(\xi)|\,|\omega(x)|,$$

$$\omega(x) = (x-0)^2 (x-1)^2 (x-2)^2$$

$$f'(x) = -\frac{1}{(x+1)^2}, \quad f''(x) = \frac{2}{(x+1)^3},$$

$$f'''(x) = -\frac{6}{(x+1)^4}, \dots, \quad f^{(6)}(x) = \frac{6!}{(x+1)^7}$$

$\boxed{5}$

Ocenimo:

$$\max_{x\in[0,2]} |f(x) - p_5(x)| \le \frac{1}{6!} \max_{\xi\in[0,2]} |f^{(6)}(\xi)| \max_{x\in[0,2]} |\omega(x)|$$

$$\max_{\xi\in[0,2]} |f^{(6)}(\xi)| \le \frac{6!}{(0+1)^7} = 6!$$

10

$$\omega'(x) = 2x(x-1)^2(x-2)^2 + x^2\, 2(x-1)(x-2)^2$$

$$+\, x^2 (x-1)^2\, 2(x-2) = 0$$

$$2x(x-1)(x-2)\Big(\overbrace{(x-1)(x-2) + x(x-2) + x(x-1)}^{\tilde{\omega}(x)}\Big) = 0$$

$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 2$ – minimumi

$$\tilde{\omega}(x) = x^2 - 3x + 2 + x^2 - 2x + x^2 - x = 0$$

10

$$3x^2 - 6x + 2 = 0 \qquad \tilde{x}_{4,5} = \frac{2 \pm \sqrt{4 - \frac{8}{3}}}{2} = \frac{2 \pm \frac{4}{\sqrt{3}}}{2}$$

$$x^2 - 2x + \frac{2}{3} = 0 \quad \boxed{5}$$

5

$$\max_{x\in[0,2]} |\omega(x)| = |\omega(\tilde{x}_4)| = \frac{4}{27} \Rightarrow \max_{x\in[0,2]} |f(x) - p_5(x)| \le \frac{4}{27} = 0{,}148$$

4 $\quad y'' = -xy, \quad y(0) = y'(0) = 1$

$y_1 = y \qquad y_1' = y_2$

$y_2 = y' \qquad y_2' = -x y_1$

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow Y' = \begin{bmatrix} 0 & 1 \\ -x & 0 \end{bmatrix} Y$$ 10

$$Y^{(r+1)} = Y^{(r)} + h \begin{bmatrix} 0 & 1 \\ -x_{r+1} & 0 \end{bmatrix} Y^{(r+1)}$$

$$Y^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow Y^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1/2 & 0 \end{bmatrix} Y^{(1)}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1/2 \\ -1/4 & 0 \end{bmatrix} \right) Y^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & -1/2 \\ 1/4 & 1 \end{pmatrix} Y^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ 10

$$\left.\begin{aligned} y_1^{(1)} - \tfrac{1}{2} y_2^{(1)} &= 1 \quad / \cdot 2 \\ \tfrac{1}{4} y_1^{(1)} + y_2^{(1)} &= 1 \end{aligned}\right\} + \quad \frac{9}{4} y_1^{(1)} = 3$$

5 $\quad \boxed{y_1^{(1)} = \frac{4}{3}}$

$$y_2^{(1)} = 1 - \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{3} = \frac{2}{3}$$

$$y'(0.5) \approx \frac{2}{3} \approx 0.66\cancel{7}$$