1. Prove that
$$\prod_{k=1}^{n} \frac{1}{q^k t} = \sum_{k \geq 0} q^k \binom{n+k-1}{\underline{k}} t^k \, .$$

2. Determine the number of North-East paths from $(0,0)$ to (k,j).

   Prove that
$$\sum_{p\colon \text{N-E path}} q^{pl(p)} = \binom{k+j}{\underline{k}},$$
   where $pl(p)$ is the number of 1×1 squares left/above of the path p.

3. Say that a permutation $\pi \in S_{2n}$ has property $\mathcal{P}$ if for some $i \in [2n]$, $|\pi(i) - \pi(i+1)| = n$, where $i+1$ is taken modulo $2n$. Show that, for each n, there are more permutations with property $\mathcal{P}$ than without it.

   Hint: You can use the inequality
$$\left| \bigcup_{i=1}^{m} A_i \right| \geq \sum_{i=1}^{m} |A_i| - \sum_{\{i,j\} \subset [m]} |A_i \cap A_j|.$$

   Possible hint: Consider the sets $A_i = \{\pi \; ; \; |\pi(i) - \pi(i+1)| = n\}$.