Kompleksna analiza (M2113) **Name:** ____________________

First semester

05/02/2018

Time Limit: 180 Minutes

Notes:

- Attempt all questions and write your answers (in *English*) in the space provided below that question.
- If there is insufficient space below a question, then use the space to the left of that question, indicating clearly which question you are answering.
- Only work written in this question and answer booklet will be marked.
- Examination materials must not be removed from the examination room.

Materials

1. One double-sided A4 page of notes,
2. Calculators are not permitted.

Grade Table (for TA's use only)

Question	Points	Score
1	16	
2	10	
3	12	
4	16	
5	7	
6	14	
7	10	
Total:	85	

Do not commence writing until instructed to do so

1. (16 points) Let Ω be an open subset of $\mathbb{C}$ and let u be a $\mathscr{C}^2$ real-valued function on Ω.
   (a) (1 point) State what it means for u to be *harmonic*.

   (b) (1 point) State what it means for u to be *subharmonic*.

   (c) (6 points) Decide whether the following functions are harmonic, subharmonic or both.
      1. $u(x,y) = xy - x^3 + 3xy^2$,
      2. $u(x,y) = x^4 + 6x^2y^2 - y^4$.

   (d) (4 points) Let $f \in \mathscr{C}^2(\Omega)$. Show that $\Delta f = \frac{1}{4}\frac{\partial^2 f}{\partial z \partial \bar{z}}$,

(e) (4 points) Hence show that if f is holomorphic and zero-free, then $\log|f|$ is harmonic.

2. (10 points) (a) (3 points) Compute $\int\limits_C \frac{e^{z^2}}{z^2-1} dz$, where C is the circle centred at 1 of radius 1, oriented counterclockwise.

(b) (3 points) Find an integral formula for a solution of the equation $\frac{\partial f}{\partial \bar{z}} = y + \exp(z)$ on $\mathbb{D}$. *Do not try to evaluate this integral!*

(c) (4 points) Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map. We say that z is a fixed point of f if $f(z) = z$. Show that every holomorphic map $f : \mathbb{D} \to \mathbb{D}$ which has two fixed points is constant.

3. (12 points) Let Ω be an open subset of $\mathbb{C}$ and let $\mathcal{F} \subset \mathscr{O}(\Omega)$ be a family of holomorphic functions on Ω.

   (a) (2 points) State what it means for $\mathcal{F}$ to be a *normal family.*

   (b) (2 points) State what it means for $\mathcal{F}$ to be *equibounded on every compact set* $K \subset \Omega$.

   (c) (2 points) State *Montel's theorem,*

   (d) (4 points) Prove that the family

$$\mathcal{F} = \left\{ \sum_{n=1}^{\infty} a_n \sin(nz) : |a_n| < \frac{1}{n!}, \forall n \in \mathbb{N} \right\}$$

   is normal in $\mathbb{C}$.

(e) (2 points) Prove that the family $\mathcal{F} = \{f \in \mathscr{O}(\mathbb{D}) : f(0) = 0, f'(0) = \cdots = f^n(0) = 1\}$ is *not* normal in $\mathbb{D}$ for any $n \in \mathbb{N}$.

4. (16 points) Suppose that K is a compact subset of a domain $\Omega \subset \mathbb{C}$.

   (a) (2 points) Write down the definition of the $\mathscr{O}(\Omega)$-hull of K.

   (b) (4 points) Show that if $\Omega \subset \Omega'$ are domains and $K \subset \Omega \cap \Omega'$, then $\hat{K}_{\mathscr{O}(\Omega)} \subset \hat{K}_{\mathscr{O}(\Omega')}$.

   (c) (4 points) Give an example of a compact set K and domains $\Omega \subset \Omega'$ such that $\hat{K}_{\Omega} \neq \hat{K}_{\Omega'}$. *Briefly* justify your answer.

(d) (6 points) Construct a sequence of polynomials that converges (pointwise) to $f : \mathbb{C} \to \mathbb{C}$,

$$f(x+iy) = \begin{cases} 1 & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -1 & \text{if } y < 0. \end{cases}$$

5. (7 points) Let Ω be an open subset of $\mathbb{C}$.

   (a) (2 points) State what it means for a sequence of compact sets $(K_j)_{j\in\mathbb{N}}$ to be a *normal exhaustion* for Ω

   (b) (2 points) Give an example of a normal exhaustion for $\mathbb{C}$.

   (c) (3 points) By adapting your answer to part (b), give a normal exhaustion for a general domain $\Omega \subset \mathbb{C}$.

6. (14 points) (a) (2 points) State the definition of the Schwarz class $\mathscr{S}$,

   (b) (4 points) State Koebe's 1/4-theorem and (small) Picard's theorem

   (c) (4 points) Show that there is a solution to the equation $e^{2z} - 1 = \pi/8$ in $\mathbb{D}$.

   (d) (4 points) Show that there is a solution to the equation $e^{\sin(z)} = \pi$ in $\mathbb{C}$.

7. (10 points) (a) (4 points) Prove that the function $f(z) = \prod_{n=0}^{\infty}(1 + z^{2^n})$ is holomorphic on $\mathbb{D}$.

(b) (6 points) Show that $\prod_{n=1}^{\infty}(1 + z^{2^n}) = \frac{1}{1 - z^2}$ for $z \in \mathbb{D}$.