Kompleksna analiza (M2113) **Name:** ____________________
25/04/2018
Time Limit: 180 Minutes

Notes:

- Attempt all questions and write your answers (in *English*) in the space provided below that question.
- If there is insufficient space below a question, then use the space to the left of that question, indicating clearly which question you are answering.
- Only work written in this question and answer booklet will be marked.
- Examination materials must not be removed from the examination room.

Materials

1. One double-sided A4 page of notes,
2. Calculators are not permitted.

Grade Table (for TA's use only)

Question	Points	Score
1	16	
2	10	
3	14	
4	7	
5	11	
6	12	
7	15	
Total:	85	

Do not commence writing until instructed to do so

1. (16 points) Let Ω be an open subset of $\mathbb{C}$ and let u be a $\mathscr{C}^2$ real-valued function on Ω.

   (a) (1 point) State what it means for u to be *harmonic*.

   (b) (1 point) State what it means for u to be *subharmonic*.

   (c) (6 points) Decide whether the following functions are harmonic, subharmonic or both.

      1. $u(x, y) = x^3 - 2xy^2 - y^2x$,
      2. $u(x, y) = x^4 + 6x^2y^2 - y^4$.

(d) (4 points) Let $f \in \mathscr{C}^2(\Omega)$. Show that $\Delta f = 4\frac{\partial^2 f}{\partial z \partial \bar{z}}$,

(e) (4 points) Hence show that if f is holomorphic and zero-free, then $\log|f|$ is harmonic.

2. (10 points) (a) (3 points) Compute $\int\limits_C \frac{e^{z^2-1}}{z^2+z-2}dz$, where C is the circle centred at 1 of radius 1, oriented counterclockwise.

(b) (3 points) Find an integral formula for a solution of the equation $\frac{\partial f}{\partial \bar{z}} = x + \exp(z^2)$ on $\mathbb{D}$. *Do not try to evaluate this integral!*

(c) (4 points) Suppose Ω is a simply connected domain in $\mathbb{C}$ not equal to $\mathbb{C}$. Show that if $f : \Omega \to \Omega$ is holomorphic and $a \in \Omega$ is a fixed point of f, then $|f'(a)| \leq 1$.

3. (14 points) Let Ω be an open subset of $\mathbb{C}$ and let $\mathcal{F} \subset \mathscr{O}(\Omega)$ be a family of holomorphic functions on Ω.

   (a) (2 points) State what it means for $\mathcal{F}$ to be a *normal family.*

   (b) (2 points) State what it means for $\mathcal{F}$ to be *equibounded on every compact set* $K \subset \Omega$.

   (c) (2 points) State *Montel's theorem*,

   (d) (4 points) Prove that the family

$$\mathcal{F} = \left\{ \frac{z}{z-a} : |a| \geq 1 \right\}$$

   is normal in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

(e) (4 points) Give an example of a domain $\Omega \subset \mathbb{C}$ and a family of maps in $\mathcal{F} \subset \mathscr{O}(\Omega)$ which is not normal.
You should explain why the example you have given is not a normal family.

4. (7 points) Let Ω be an open subset of $\mathbb{C}$.

   (a) (2 points) State what it means for a sequence of compact sets $(K_j)_{j\in\mathbb{N}}$ to be a normal exhaustion for Ω

   (b) (2 points) Give an example of a normal exhaustion for $\mathbb{C}$.

   (c) (3 points) By adapting your answer to part (b), give a normal exhaustion for a general domain $\Omega \subset \mathbb{C}$.

5. (11 points) Suppose that K is a compact subset of a domain $\Omega \subset \mathbb{C}$.

   (a) (2 points) Write down the definition of the $\mathscr{O}(\Omega)$-hull of K.

   (b) (3 points) Prove that if K is compact, then $\hat{K}$ is bounded.

   (c) (3 points) Let $K = \{z \in \mathbb{C} : \frac{1}{2} \leq |z| \leq 1\}$. Prove that $\widehat{K} = \overline{\mathbb{D}}$.

   (d) (3 points) Prove that for every holomorphic function $f : \mathbb{D} \to \mathbb{C}$, there is a sequence $(f_n)_{n\in\mathbb{N}}$ of entire functions $f_n : \mathbb{C} \to \mathbb{C}$ such that $f_n|_{\mathbb{D}} \to f$ uniformly on compact subsets of $\mathbb{D}$.

6. (12 points) (a) (2 points) State the definition of the Schwarz class $\mathscr{S}$,

(b) (4 points) Let $f : \mathbb{D} \to \mathbb{C}$,

$$f(z) = \frac{1}{10}(e^{10z} - 1).$$

Show that $f(0) = 0$, $f'(0) = 1$ and f omits the value $-1/10$. Explain why this *does not* contradict Koebe's 1/4-theorem.

(c) (2 points) State (small) Picard's theorem.

(d) (4 points) Let f be an entire non-constant function that satisfies the functional equation

$$f(1-z) = 1 - f(z)$$

for all $z \in \mathbb{C}$. Show that $f(\mathbb{C}) = \mathbb{C}$.

7. (15 points) (a) (8 points) Suppose $f : \mathbb{C} \to \mathbb{C}$ is a non-constant holomorphic function satisfying

$$f(2z) = (1 - 2z)f(z)$$

for $z \in \mathbb{C}$. Show that f has a simple zero at 2^k for $k = 0, 1, 2, \ldots$ and that f is non-zero elsewhere.

Hint: By analyticity theorem $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *for all* $z \in \mathbb{C}$, where $a_n = f^{(n)}(0)/n!$

(b) (4 points) Find a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ satisfying the equation of Part (a).

Hint: Find an infinite product $\prod_{k=0}^{\infty}(1 + a_n(z))$ *converging uniformly on compact subsets of* $\mathbb{C}$ *with the same set of zeros as* f.

(c) (3 points) Hence determine *all* holomorphic solutions $f : \mathbb{C} \to \mathbb{C}$ to the above functional equation.