Numerične metode-fiziki, 10. 9. 2019
Čas pisanja je 90 minut. Možno je doseči 100 točk. Uporabljate lahko pisalo, kalkulator in dva lista formata A4 z zapiski!

1	2	3	4	Σ

Ime in priimek

Vpisna številka

Naloga 1 [25 točk]

Izračunajte LU razcep z delnim pivotiranjem za matriko

$$A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}.$$

Nato s pomočjo tega razcepa izračunajte $\det A$.

$$\begin{matrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{matrix} \longrightarrow \begin{matrix} 2 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & 2 & -1 & 0 \\ -1 & 0 & 1 & 2 \end{matrix} \xrightarrow{\text{elim.}} \begin{matrix} 2 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 1/2 & 5/2 & -1 & -1/2 \\ -1/2 & -1/2 & 1 & 5/2 \end{matrix} \longrightarrow \begin{matrix} 2 & -1 & 0 & 1 \\ 1/2 & 5/2 & -1 & -1/2 \\ 0 & 1 & 2 & -1 \\ -1/2 & -1/2 & 1 & 5/2 \end{matrix}$$

$$\xrightarrow{\text{elim.}} \begin{matrix} 2 & -1 & 0 & 1 \\ 1/2 & 5/2 & -1 & -1/2 \\ 0 & 2/5 & 12/5 & -4/5 \\ -1/2 & -1/5 & 4/5 & 12/5 \end{matrix} \xrightarrow{\text{elim}} \begin{matrix} 2 & -1 & 0 & 1 \\ 1/2 & 5/2 & -1 & -1/2 \\ 0 & 2/5 & 12/5 & -4/5 \\ -1/2 & -1/5 & 1/3 & 8/3 \end{matrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 2/5 & 1 & 0 \\ -1/2 & -1/5 & 1/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 5/2 & -1 & -1/2 \\ 0 & 0 & 12/5 & -4/5 \\ 0 & 0 & 0 & 8/3 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$PA = LU \Rightarrow \det P \cdot \det A = \det L \cdot \det U$$

$$1 \cdot \det A = 1 \cdot 32$$

$$\det A = 32$$

Naloga 2 [25 točk]

Določite parametra $\alpha, \beta \in \mathbb{R}$ v iterativni metodi

$$x_{n+1} = \alpha x_n + \frac{\beta}{x_n}, \quad n = 0, 1, \ldots,$$

tako, da bo metoda za dovolj dober začetni približek x_0 konvergirala k $\sqrt{a}$, $a > 0$, s čim večjim redom konvergence. Kakšen je ta red?

$$g(x) = \alpha x + \frac{\beta}{x}$$

① $g(\sqrt{a}) = \sqrt{a}$ (negibna točka)

$$\Downarrow$$

$$\alpha\sqrt{a} + \frac{\beta}{\sqrt{a}} = \sqrt{a} \quad / \sqrt{a}$$

$$\alpha a + \beta = a \Rightarrow \underline{\underline{\beta = a - \alpha a}}$$

② $g'(\sqrt{a}) = 0$ (red nad 1)

$$g'(x) = \alpha - \frac{\beta}{x^2} \Rightarrow g'(\sqrt{a}) = \alpha - \frac{\beta}{a} = 0$$

$$\Rightarrow \underline{\underline{\beta = \alpha \cdot a}} \Rightarrow \alpha a = a - \alpha a$$

$$\boxed{\alpha = \frac{1}{2}} \Rightarrow \boxed{\beta = \frac{1}{2} a}$$

$$g''(x) = \frac{2\beta}{x^3} \Rightarrow g''(\sqrt{a}) = \frac{\cancel{2} \cdot \frac{1}{\cancel{2}} a}{(\sqrt{a})^3} = \frac{1}{\sqrt{a}} \neq 0$$

$\Rightarrow$ red je $\underline{\underline{2}}$

Naloga 3 [25 točk]

Vrednosti **in prve odvode** funkcije $f(x) = \sqrt{x}$ interpolirate v točkah $x_0 = 1$ in $x_1 = 4$. Izračunajte ustrezni interpolacijski polinom p in čim bolje ocenite napako

$$\max_{x\in[1,4]} |f(x) - p(x)|\,.$$

x	$f(x)$			
1	1			
		$1/2$		
1	$1/2$		$-1/18$	
		$1/3$		$1/108$
4	2		$-1/36$	
		$1/4$		
4	$1/4$			

odvoda

$\Downarrow$

$$p(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{18}(x-1)^2 + \frac{1}{108}(x-1)^2(x-4)$$

$$\omega(x) = (x-1)^2(x-4)^2$$

$$\max_{x\in[1,4]} |f(x) - p(x)| = \max_{x\in[1,4]} \left|\frac{1}{4!} f^{(IV)}(\xi)\, \omega(x)\right|$$

$$f^{(IV)}(x) = -\frac{15}{16} x^{-7/2} \qquad \omega'(x) = 2(x-1)(x-4)(2x-5)$$

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 4, \quad \tilde{x}_3 = 5/2$$

$$\max_{x\in[1,4]} |f(x) - p(x)| \leq \frac{1}{4!}\,\frac{15}{16}\,|\omega(5/2)| \doteq 0{,}1978$$

Naloga 4 [25 točk]

Višino delca ob času t opisuje brezdimenzijska funkcija $y(t)$. Pospešek v vertikalni smeri se izraža kot

$$\ddot{y}(t) = -1 - \dot{y}(t)^2, \quad t \geq 0.$$

Ob času $t = 0$ se delec nahaja na višni $y(0) = 10$ in ima hitrost $\dot{y}(0) = 0$. Z eksplicitno Eulerjevo metodo

$$y_{n+1} = y_n + \Delta t\, f(t_n, y(t_n), \dot{y}(t_n)), \quad n = 0, 1, \ldots,$$

za sistem diferencialnih enačb ocenite višino, hitrost in pospešek delca ob času $t = 1$. Za korak vzemite $\Delta t = 1/2$.

Prevedba na sistem: $y_1 = y,\ y_2 = \dot{y} \Rightarrow$

$$\begin{aligned}\dot{y}_1 &= y_2 \\ \dot{y}_2 &= -1 - y_2^2\end{aligned} \Rightarrow Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ in } F(t, Y) = \begin{bmatrix} y_2 \\ -1 - y_2^2 \end{bmatrix}$$

$$Y_1 = Y_0 + \frac{1}{2} F(\overset{\overset{0}{\shortparallel}}{t_0}, Y_0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ -1 - 0^2 \end{bmatrix} = \begin{bmatrix} 10 \\ -1/2 \end{bmatrix}$$

$$Y_2 = Y_1 + \frac{1}{2} F(\overset{\overset{1/2}{\shortparallel}}{t_1}, Y_1) = \begin{bmatrix} 10 \\ -1/2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1/2 \\ -1 - 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 9.75 \\ -1.125 \end{bmatrix} \Rightarrow \begin{aligned} y(1) &\approx 9.75 \\ \dot{y}(1) &\approx -1.125 \\ \ddot{y}(1) &= -1 - \dot{y}(1)^2 \approx -2.2656 \end{aligned}$$