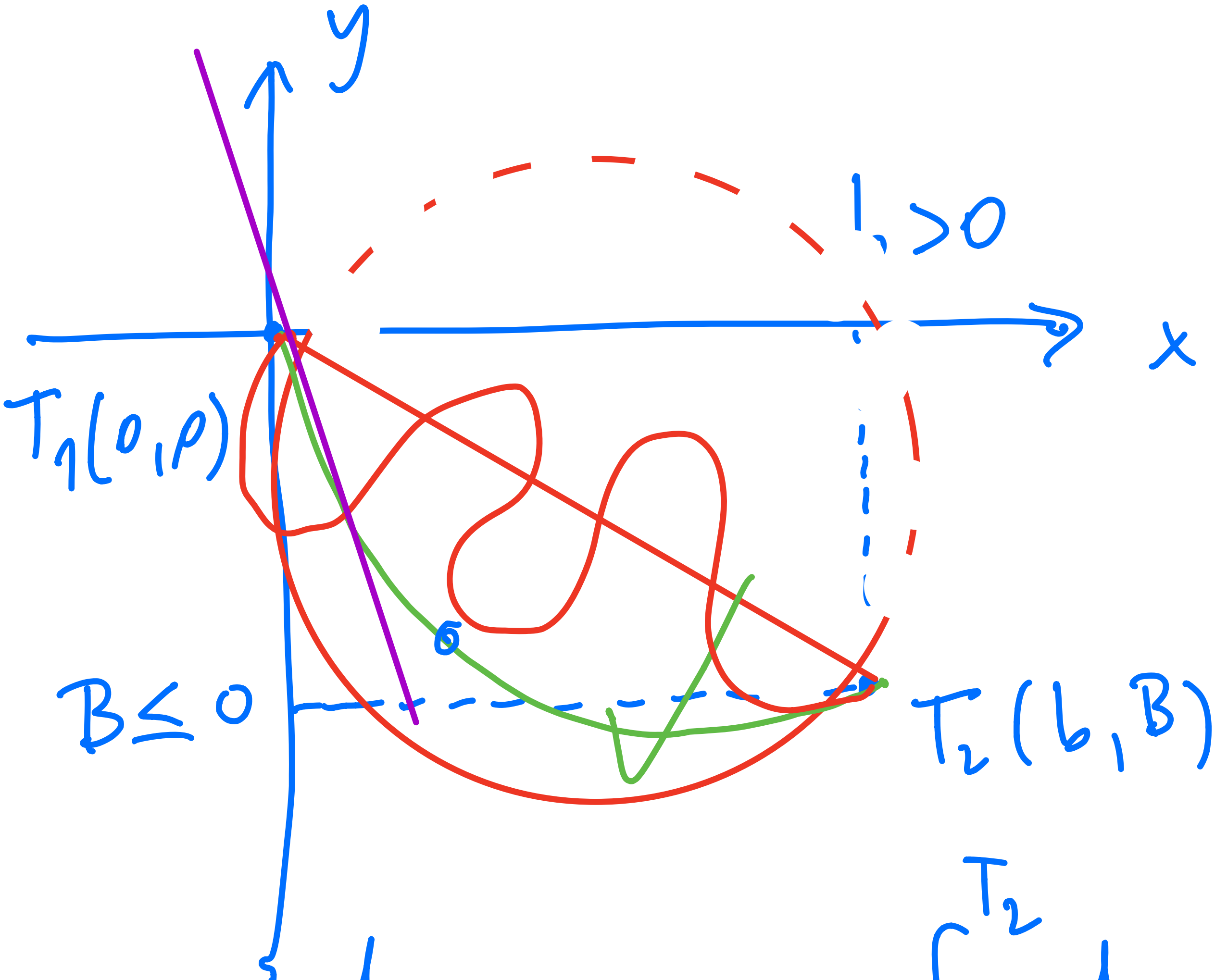
Brahistohrona!



$$v = \frac{ds}{dt} \Rightarrow t = \int_{T_1}^{T_2} \frac{ds}{v} \; ; \; \underline{\underline{min}}$$

$$\Delta W_k = \Delta W_p \; ; \; v_0 = 0 \text{ (začetek)}$$

$$\frac{1}{2} \cancel{m} v^2 = \cancel{m} g \cdot 0 - \cancel{m} g y$$

$v = \sqrt{-2gy}$; $ds = \sqrt{1+y'^2}\,dx$

$$\int_{T_1}^{T_2} \frac{\sqrt{1+y'^2}}{\sqrt{-2gy}}\,dx = \min$$

$$\boxed{y = y(x)}$$

$$\int_a^b f(x, y, y')\,dx \text{ ima}$$

min ⇐ reša E. L. enačba

$$f(x, y, y') - y'\frac{\partial f}{\partial y'}(x, y, y') = k_1,$$

V našem primeru je

$$f(x, y, y') = \sqrt{\frac{1+y'^2}{-2gy}}$$

$$\ldots \quad \boxed{-y(1+y'^2) = k^2}, \quad k\text{-konst}$$

Nastavek: $y = -k^2 \sin^2 t$

$$\boxed{dy = -k^2\, 2 \sin t \cos t} \Rightarrow$$

$$\left(\frac{dy}{dx}\right)^2 = -\frac{k^2}{y} - 1 = \left(\frac{\cos t}{\sin t}\right)^2$$

$$\boxed{\frac{dy}{dx} = -\frac{\cos t}{\sin t}}$$

$$dx = \frac{dx}{dy}\frac{dy}{dt}\,dt = \frac{1}{y'}\frac{dy}{dt}\,dt$$

$$dx = 2k^2 \sin^2 t \, dt \quad / \int$$

$$x = k^2\left(t - \frac{\sin 2t}{2}\right) + C$$

Upostawiamo: $x(0) = 0$

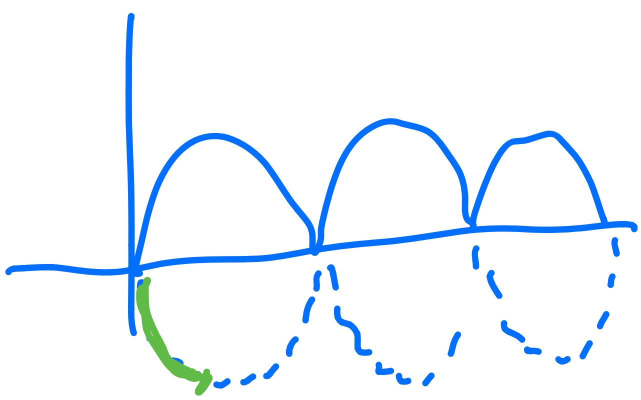
$\Rightarrow C = 0$; $\theta = 2t$:

$$x(\theta) = \frac{1}{2}k^2(\theta - \sin\theta)$$

$$y(\theta) = -\frac{1}{2}k^2(1 - \cos\theta)$$

$\theta \in [0, ?]$

CIKLOIDA!



Določiti je treba še k

in interval $\theta \in [0, \frac{\theta^*}{2}]$

$x(0) = 0$ ✓, $y(0) \doteq 0$ ✓

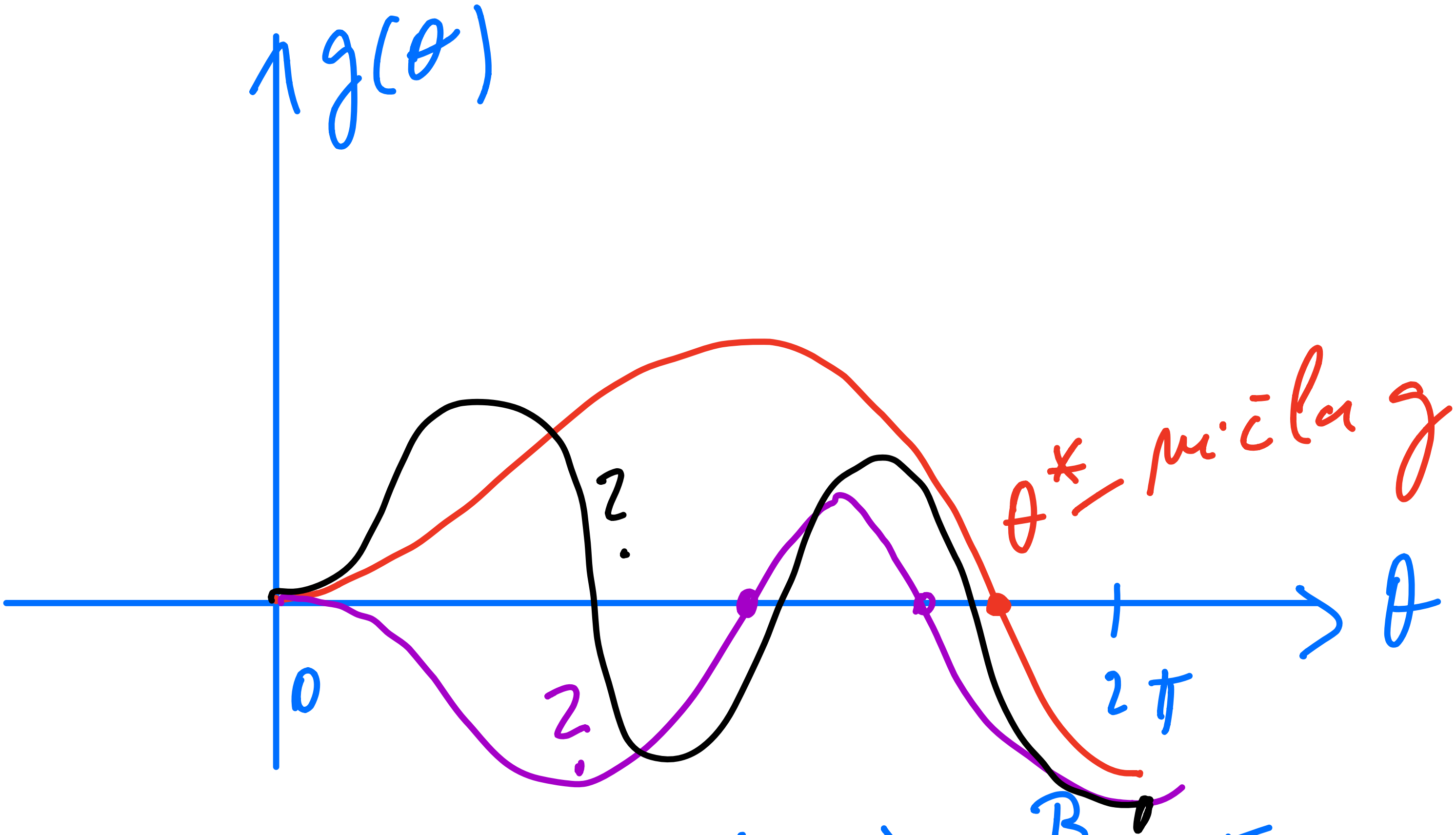
$$x(\theta^*) = b = \frac{1}{2}k^2(\theta^* - \sin\theta^*)$$

$$y(\theta^*) = B = -\frac{1}{2}k^2(1 - \cos\theta^*)$$

Sistem dveh (nelin.) enačb za k in $\theta^* = \theta$.

$$\frac{y(\theta^*)}{x(\theta^*)} = \frac{-\frac{1}{2}k^2(1-\cos\theta^*)}{\frac{1}{2}k^2(\theta^* - \sin\theta^*)} = \frac{B}{b}$$

... $\boxed{g(\theta) = 1 - \cos\theta + \frac{B}{b}(\theta - \sin\theta) = 0}$



$g(0) = 0$, $g(2\pi) = \frac{B}{b} 2\pi < 0$

$g'(\theta) = \sin\theta + \frac{B}{b}(1 - \cos\theta)$

$g'(0) = 0$

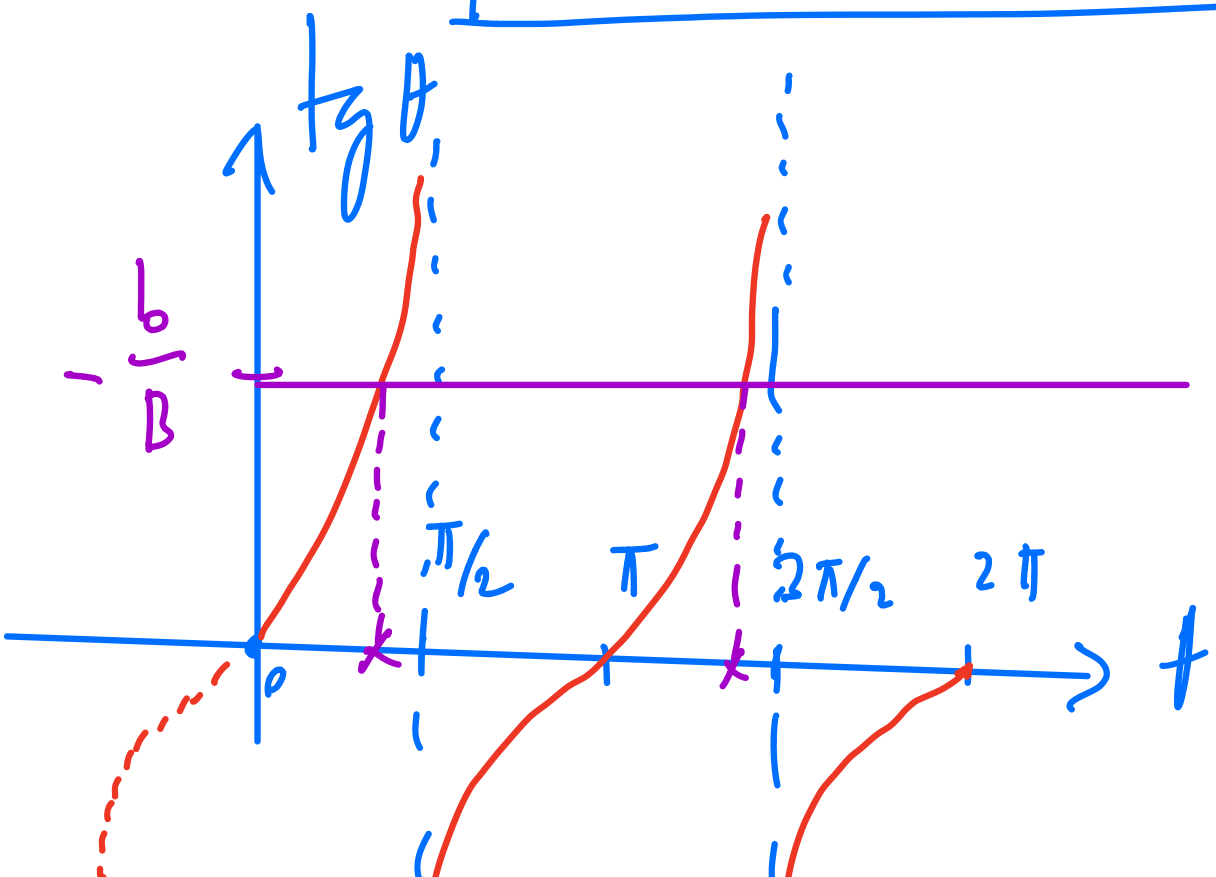
$g''(\theta) = \cos\theta + \frac{B}{b}\sin\theta$

$g''(0) = 1 > 0 \Rightarrow$ wypukła w okolicy 0!

Če želimo dokazati, da ima g natanko eno ničlo na $[0, 2\pi)$, je dovolj videti, da ima g'' natanko ~~eno~~ dve ničli na $[0, 2\pi)$!

$$g''(\theta) = \cos\theta + \frac{B}{b}\sin\theta = 0$$

$$\boxed{\operatorname{tg}\theta = -\frac{b}{B}} > 0$$



Ker ima g'' dve ničli na $[0, 2\pi)$ in g' kvečjemu dve ničli na $[0, 2\pi)$ in posledično g natanko eno ničlo $\underline{\underline{\theta^*}}$ na $(0, 2\pi)$.

$\Rightarrow$ g ima za vsak $\frac{B}{b}$ natanko eno ničlo θ^{*b} na $(0, 2\pi)$.

Enačbo $g(\theta) = 0$ v Matlabu rešite z <u>fzero</u>.

Čas potovanja kroglice

$$\int_0^b \sqrt{\frac{1+y'(x)^2}{-2g\,y(x)}}\,dx$$

$$x(\theta) = \frac{1}{2}k^2(\theta - \sin\theta)$$

$$y(\theta) = -\frac{1}{2}k^2(1-\cos\theta)$$

$$\int_0^{\theta^*} \sqrt{\frac{1+\left(\frac{dy}{dx}(\theta)\right)^2}{-2g\,y(\theta)}}\,dx(\theta)$$

$$= \int_0^{\theta^*} \sqrt{\frac{1+\left(\frac{-\cancel{\frac{1}{2}}\cancel{k^2}\sin\theta}{\cancel{\frac{1}{2}}\cancel{k^2}(1-\cos\theta)}\right)^2}{+\cancel{2}g\left(+\cancel{\frac{1}{2}k}^2(1-\cos\theta)\right)}}\;.$$

$$\cdot \frac{1}{2} k^2 (1-\cos\theta)\, d\theta$$

$$= \int_0^{\theta^*} \sqrt{\frac{1+\dfrac{\sin^2\theta}{(1-\cos\theta)^2}}{g k^2 (1-\cos\theta)}} \cdot$$

$$\frac{1}{2} k^2 (1-\cos\theta)\, d\theta$$

$$= \frac{1}{2} k^{\cancel{2}} \frac{1}{\cancel{k}\sqrt{g}} \int_0^{\theta^*} \sqrt{\frac{1-2\cos\theta+\overbrace{\cos^2\theta+\sin^2\theta}^{1}}{(1-\cos\theta)^3}}$$

$$\cdot (1-\cos\theta)\, d\theta$$

$$= \frac{k}{2\sqrt{g}} \sqrt{2} \int_0^{\theta^*} \sqrt{\frac{\cancel{1-\cos\theta}}{(1-\cos\theta)^{\cancel{3}\,2}}}.$$

$$(1-\cos\theta)\,d\theta$$

$$= \frac{k}{\sqrt{2g}} \int_0^{\theta^*} \frac{1-\cancel{\cos\theta}}{1-\cancel{\cos\theta}}\,d\theta$$

$$= \frac{k}{\sqrt{2g}}\,\theta^*$$