The axiom of choice

Recap:

- The well-orderable sets are exactly the sets whose cardinalities are alephs
- If X, Y are well-orderable then so are:
  - $X+Y$
  - $X \times Y$
  - any subset of X
  - any image of X (along a function with domain X)

We have not shown that if X, Y are well-orderable then so are Y^X and $\mathcal{P}X$.

In fact its not possible to prove this from our axioms.

The axiom of choice will assert that all sets are well-orderable, so it circumvents all concerns about well-orderability

Let S be a set of sets. A <u>choice function</u> on S is a function f with domain S that satisfies:

$$\text{for all } X \in S, \quad X \neq \emptyset \quad \Rightarrow f(X) \in X.$$

<u>Theorem</u> The following are equivalent for a set X.

(1) X is well-orderable

(2) $\mathcal{P}(X)$ has a choice function.

<u>Proof</u>

<u>(1) ⇒ (2)</u> Suppose $\prec$ is a well-ordering of X

Then the following defines a choice function on $\mathcal{P}X$

$$Y \longmapsto \begin{cases} \text{the least element of } Y \text{ under } \prec & \text{if } Y \neq \emptyset \\ \emptyset \quad \text{(say)} & \text{if } Y = \emptyset \end{cases}$$

<u>(2) ⇒ (1)</u> Suppose g is a choice function on $\mathcal{P}X$

Let e ("end") be some element of $\underline{U}$ not in X.

Define $y_{(-)} : \underline{\mathrm{Ord}} \longrightarrow X \cup \{e\}$ by transfinite recursion

$$y_\alpha = \begin{cases} g(X - \{y_\beta \mid \beta < \alpha\}) & \text{if } X - \{y_\beta \mid \beta < \alpha\} \neq \emptyset \\ e & \text{otherwise} \end{cases}$$

Then:

- $\{\alpha \mid y_\alpha \in X\}$ is an initial segment of $\underline{Ord}$.

  I.e. $y_\alpha \in X$ and $\beta \leq \alpha$ implies $y_\beta \in X$. Exercise.

- If $y_\alpha = y_\beta \in X$ then $\alpha = \beta$. Exercise.

- There exists α s.t. $y_\alpha = e$. Exercise.

Let α be the least ordinal s.t. $y_\alpha = e$

Then $y_{(-)} : \downarrow\alpha \to X$ is a bijection.

Hence X is well-orderable.

$\square$

Remark: we used transfinite recursion in the following form

<u>Theorem</u> (Transfinite recursion, uniform version)

Given any class $\underline{X}$ and class function $\underline{P} : \mathcal{P}(\underline{X}) \to \underline{X}$, there exists a unique class function $\underline{f} : \underline{Ord} \to \underline{X}$ satisfying

$$\underline{f}(\alpha) = \underline{P}\,\{\underline{f}(\beta) \mid \beta < \alpha\}.$$

Exercise: prove this directly, or derive from previous transfinite recursion theorem

Which sets of sets have choice functions?

<u>Proposition</u> Every finite set of sets S has a choice function.

<u>Proof</u> By induction on $|S|$. Details: exercise.

We cannot prove in general that an infinite set of sets, even a countably infinite set of sets, has a choice function.

We know the following sets of sets have choice functions:

- Finite sets of sets
- $\mathcal{P}(X)$ where X is well-orderable

  (In particular $\mathcal{P}(\downarrow\alpha)$ for $\alpha \in \underline{\mathrm{Ord}}$)

We cannot prove:

- $\mathcal{P}(\mathbb{R})$ have a choice function
- Ely $\mathbb{R}$ is well-orderable

The **axion of choice (AC)**: Every set of sets has a choice function.

For the moment we do not add AC to our list of axioms.

Instead we explore consequences of AC and equivalent statements.

Theorem t.f.a.e.

(1) The axiom of choice

Every set of sets has a choice function.

(2) The well-ordering principle

Every set is well-orderable

(3) Zorn's lemma

If every chain in a partially ordered set has an upper bound then the partial order has a maximal element.

Explanation of terminology in Zorn's lemma

- A <u>chain</u> in a partially ordered set $(X, \le)$ a subset $C \subseteq X$ that is totally ordered by $\le$ (i.e. if $x, y \in C$ then $x \le y$ or $y \le x$)
- A <u>maximal element</u> is $x \in X$ that satisfies: there is no $y \in X$ with $y > x$.
- An <u>upper bound</u> of $C \subseteq X$ is any $z \in X$ that satisfies $\forall y \in C\ (y \le z)$
- z is a <u>strict upper bound</u> for C if $\forall y \in C\ (y < z)$

Proof of Theorem

$(1) \Leftrightarrow (2)$ follows from earlier theorem.

We prove $(1) \Leftrightarrow (3)$.

<u>$(1) \Rightarrow (3)$</u>

Let $(X, \le)$ be a partial order in which every chain has an upper bound.

Let g be a choice function for $\mathcal{P}(X)$.

Let e be an element of $\underline{U} - X$

Define $y_{(-)} : \underline{\mathrm{Ord}} \to X$ by transfinite recursion:

$$y_\alpha = \begin{cases} g(x \in X \mid x \text{ is a strict upper bound of } \{y_\beta \mid \beta < \alpha\}) \\ \qquad \text{if such a strict upper bound exists} \\ e \quad \text{otherwise} \end{cases}$$

We observe that

- if $\beta < \alpha$ and $y_\alpha < X$ then $y_\beta \in X$ and $y_\beta < y_\alpha$
- there exists α s.t. $y_\alpha = e$.

Let α be least ordinal s.t. $y_\alpha = e$

Then $\{y_\beta \mid \beta < \alpha\}$ is a chain in X

So this chain has an upper bound $y \in X$.

That is, $\forall \beta < \alpha \quad y_\beta \leq y$.

Since $y_\alpha = e$, y is $\underline{\text{not}}$ a strict upper bound.

Indeed no strict upper bound exists.

This means that y is a maximal element of X.

$(3) \Rightarrow (1)$

nonempty ← (without loss of generality)

Let S be a set of nonempty sets. Consider:

$$F = \{ f \mid f \text{ is a choice function on some subset } S' \subseteq S \}$$

(F is a set!)

F is partially ordered by $\subseteq$

Specifically

$$f \subseteq f' \iff \operatorname{dom}(f) \subseteq \operatorname{dom}(f') \text{ and } \forall x \in \operatorname{dom}(f) \quad f(x) = f'(x)$$

Observe that if $C \subseteq F$ is a chain then

$\bigcup C$ is a choice function $\bigcup_{f \in C} \operatorname{dom}(f)$

So $\bigcup C \in F$ is an upper bound for C.

Thus F satisfies the conditions of Zorn's Lemma.

So F has a maximal element f_0.

We show that $dom(f_0) = S$.

Suppose not.

Then there exists $x_0 \in X_0 \in S - dom(f_0)$

But then $f_0 \cup \{(X_0, x_0)\}$ is a choice function on $dom(f_0) \cup \{X_0\}$ and so in F.

Contradicting that f_0 is a maximal element.

So indeed $dom(f_0) = S$.

Hence f_0 is the required choice function on S.

$\square$

The axiom of choice (Part 2)

Two other equivalents to AC

Proposition AC is equivalent to (that

$$(\forall i \in I \;\; X_i \neq \emptyset) \Rightarrow (\prod_{i \in I} X_i) \neq \emptyset$$

holds for every set I and I-indexed family of sets $\{X_i\}_{i \in I}$.)

Reminder – an I-indexed family of sets is a function $X_{(-)} : I \to \underline{S}$ ← the class of all sets

The Cartesian product is defined by

$$\prod_{i \in I} X_i := \{ x_{(-)} \mid x_{(-)} \text{ is a function with domain } I \text{ and } \forall i \in I \;\; x_i \in X_i \}$$

Exercise: prove the proposition.

Proposition The axiom of choice is equivalent to the statement that every surjection has a section.

A *surjection* is a surjective function $r : X \to Y$ between two sets. A *section* of r is a function $s : Y \to X$ such that $r \circ s = \mathrm{id}_Y$; i.e.

for all $y \in Y$ $r(s(y)) = y$; i.e. $s(y) = r^{-1}(y)$

Exercise – prove the proposition.

(If you know category theory, the proposition states that AC is equivalent to the statement:

every epimorphism in *Set* splits.)

AC is often used in mathematics.

__But__ it also has some strange consequences.

Both points are the topic of next week's lecture.

Some applications of choice in mathematics, however, do not need the full strength of AC, and instead rely on weaker forms of choice that do not have strange consequences.

We look at two such weaker forms of choice.

$\boxed{\text{The axiom of countable choice (CC): Every countable set of sets has a choice function}}$

The axiom of dependent choice (DC)

Suppose $R \subseteq X \times X$ is a binary relation satisfying

$$\forall x \in X.\ \exists y \in X \quad xRy \qquad (\text{i.e. } (x,y) \in R).$$

Then for any $z \in X$ there exists a function $x_{(-)} : \mathbb{N} \longrightarrow X$ satisfying:

$$x_0 = z$$

$$\forall n \in \mathbb{N} \quad x_n R x_{n+1}$$

Theorem We have the following chain of implications.

$$(AC) \Rightarrow (DC) \Rightarrow (CC)$$

(Incidentally, none of the converse implications hold.)

Proof $(AC) \Rightarrow (DC)$

Suppose we have X, R and z as in the statement of (DC)

Let g be a choice function on $\mathcal{P}X$ (by AC)

We define $x_{(-)} : \mathbb{N} \to X$ using the recursion theorem:

$$x_0 = z$$

$$x_{n+1} = g\{y \in X \mid x_n R y\}$$

← the set is nonempty by assumption on R

$(DC) \Rightarrow (CC)$ We prove (CC) in the following form which is easily proved equivalent

exercise!

$\circledast$ $\boxed{\text{For any } \mathbb{N}\text{-indexed family } X_{(-)} : \mathbb{N} \to \underline{S} \text{ of nonempty sets the product } \prod_{n \in \mathbb{N}} X_n \text{ is nonempty.}}$

Suppose that $X_{(-)} : \mathbb{N} \to \underline{S}$ a family of nonempty sets.

Consider the set $X := \sum_{n \in \mathbb{N}} X_n := \{(n,x) \mid n \in \mathbb{N}, x \in X_n\}$

Define $R \subseteq X \times X$ by

$$(n,x) \, R \, (n',x') \iff n' = n+1$$

Let x_0 be some element of X_0

Define $z := (0, x_0)$

By (DC) there exists a function $f : \mathbb{N} \to X$ s.t.

$$f(0) = z$$

$$f(n) \, R \, f(n+1)$$

Then we see that

$$g(n) = x_n \text{ where } f(n) = (n, x_n)$$

this defines the required element of $\prod_{n \in \mathbb{N}} X_n$.

$\square$

Some useful consequences of CC and DC.

Proofs — exercises.

<u>Proposition A</u> (CC) ← requires this choice principle

If a set is not finite then it is infinite

<u>Proposition B</u> (CC)

A countable union of countable sets is countable

<u>Proposition C</u> (DC)

If a relation $R \subseteq X \times X$ enjoys the property:

there is no sequence $x_{(-)} : \mathbb{N} \to X$ satisfying $x_{n+1} R x_n \ \forall n \in \mathbb{N}$

then R is well-founded.