Mathematical consequences of AC

Important mathematical consequences of choice include:

- Every vector space has a basis
  (Linear algebra)
- Tychonoff's theorem: the cartesian product of a family of compact spaces is itself compact.
  (Topology)
- The Hahn-Banach theorem.
  (Functional analysis)
- The boolean prime ideal theorem
  (Boolean algebra)
- etc.

In the above cases AC serves to make the mathematics simpler than it would otherwise be without choice.

Today we look at two mathematical results in which AC has discomforting consequences:

- Vitali's theorem (measure theory)

- The Banach-Tarski theorem (set-theoretic geometry?)

Henceforth, for today's lecture, we assume AC

In fact we use AC in the following equivalent version

- Every set of equivalence classes has a set of representatives. $\circledast$

(Explicitly: If $\approx$ is an equivalence relation on a set X then there exists a subset $K \subseteq X$ such that:

for every $x \in X$ there exists a unique $z \in K$ s.t. $x \approx z$.)

Exercise: prove that $\circledast$ is equivalent to AC.

Vitali's theorem addresses the question: can we measure the "length" of an arbitrary subset $X \subseteq \mathbb{R}$, by a function $\lambda : \mathcal{P}(\mathbb{R}) \longrightarrow [0,\infty]$

E.g. consider the subset

$$[0,\tfrac{1}{4}] \cup [\tfrac{1}{2},\tfrac{5}{8}] \cup [\tfrac{3}{4},\tfrac{13}{16}] \cup [\tfrac{7}{8},\tfrac{29}{32}] \cup \cdots$$

$$\tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \cdots$$

This should have length $\tfrac{1}{2}$.

<u>Vitali's theorem</u> There is no function $\lambda : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ that satisfies:

- $\lambda(a,b) = b-a$ (where $a \leq b$)
- If $\mathcal{I} \subseteq \mathcal{P}\mathbb{R}$ is a countable set of pairwise disjoint subsets of $\mathbb{R}$ then $\lambda(\bigcup \mathcal{I}) = \sum_{X \in \mathcal{I}} \lambda(X)$. (Countable additivity)
- For any $X \subseteq \mathbb{R}$ and $d \in \mathbb{R}$, $\lambda(X+d) = \lambda(X)$. (translation invariance)

$(X+d := \{x+d \mid x \in X\})$

Before giving the proof we observe that any λ satisfying the 3 properties above also satisfies:

- $\lambda(\emptyset) = 0$
- $\lambda\{x\} = 0$
- $\lambda[a,b] = b-a \quad (a \leq b)$
- $\lambda(\mathbb{R}) = \infty$
- If $X \subseteq Y$ then $\lambda(X) \leq \lambda(Y)$

Exercise: verify that these are indeed consequences.

Proof of Vitali's theorem

Define an equivalence relation $\approx$ on $\mathbb{R}$ by

$$x \approx y \iff y - x \in \mathbb{Q}$$

This restricts to an equivalence relation on $(0,1)$

Let $K \subseteq (0,1)$ be a set of representatives for equivalence classes of $\approx$ on $(0,1)$.

It holds that: for every $x \in \mathbb{R}$, there exists a unique $z \in K$ such that $z \approx x$. $\quad (*)$

Assume we have λ satisfying the properties in the theorem.

Because $K \subseteq (0,1)$ we have $0 \leq \lambda(K) \leq 1$

Suppose $\lambda(K) = 0$

this is a disjoint union !

Then $\lambda(\mathbb{R}) = \lambda\left(\bigcup_{q \in \mathbb{Q}} K+q\right)$ by $\circledast$

$= \sum_{q \in \mathbb{Q}} \lambda(K+q)$ by countable additivity

by translation invariance $= \sum_{q \in \mathbb{Q}} \lambda(K) = \sum_{q \in \mathbb{Q}} 0 = 0$

This contradicts that $\lambda(\mathbb{R}) = \infty$. $\rightarrow\leftarrow$

We must have $\lambda(K) > 0$

Then $\lambda\left(\bigcup_{q \in \mathbb{Q} \cap [0,1]} K+q\right) = \sum_{q \in \mathbb{Q} \cap [0,1]} \lambda(K+q)$ countable additivity

$= \sum_{q \in \mathbb{Q} \cap [0,1]} \lambda(K)$ translation invariance

$= \infty$ because $\lambda(K) > 0$

However $\bigcup_{q \in \mathbb{Q} \cap [0,1]} K+q \subseteq (0,2)$ so has λ length ≤ 2. $\rightarrow\leftarrow$

$\square$

Vitali's theorem says there is no good way of assigning a notion of length to arbitrary subsets of $\mathbb{R}$.

Mathematician's need to cope with this. 3 possibilities are:

- If countable additivity is weakened to finite additivity then length functions can be proved to exist.

  However:

  - finite additivity is not strong enough for applications
  - At higher dimensions even finite additivity is not achievable

- We can weaken the length function to $\lambda : \Sigma \to [0,\infty]$ where $\Sigma \subseteq \mathcal{P}\mathbb{R}$ is a set of chosen 'measurable' sets.

  - This is the solution preferred by mathematicians.

- Drop AC

  - The existence of a length function $\lambda : \mathcal{P}\mathbb{R} \to [0,\infty]$ is consistent in the absence of AC.

The Banach-Tarski theorem (or "paradox")

Let $\mathbb{B} \subseteq \mathbb{R}^3$ be the closed unit ball.

$$\left(\mathbb{B} := \{ (x,y,z) \mid x^2+y^2+z^2 \leq 1 \} \right)$$

Then $\mathbb{B}$ can be partitioned as

$$\mathbb{B} = A_1 \cup \cdots \cup A_m \cup B_1 \cup \cdots \cup B_n$$

(a disjoint union) such that there exist proper rigid transformations (= combinations of rotations and translations) $\theta_1, \ldots, \theta_m, \phi_1, \ldots, \phi_n$ for which

$$\mathbb{B} = \theta_1 A_1 \cup \cdots \cup \theta_m A_m \quad \text{(disjoint union)}$$

$$\mathbb{B} = \phi_1 B_1 \cup \cdots \cup \phi_n B_n \quad \text{(disjoint union)}.$$

Outline proof of the Banach-Tarski theorem

(Non-examinable!)

Two important aspects of the B-T theorem are:

- It concerns finite partitions of a subset of $\mathbb{R}^3$
- It concerns the behaviour of such partitions under the action of the group SE_3 of proper rigid transformations on $\mathbb{R}^3$.

It is useful to consider these aspects at the level of generality of arbitrary group actions.

Accordingly, let G be any group acting on a set X.
(This gives us an action function $G \times X \longrightarrow X$.)

Step 1 The notion of G-equidecomposability

Subsets $A, B \subseteq X$ are said to be **G-equidecomposable** (notation $A \sim_G B$)
if they can be partitioned as

$$A = A_1 \cup \cdots \cup A_n \quad \text{(disjoint)}$$
$$B = B_1 \cup \cdots \cup B_n \quad \text{"}$$

in such a way that there exist $g_1, \ldots, g_n \in G$ s.t.

$$B_1 = g_1 A_1 \text{ and } \ldots \text{ and } B_n = g_n A_n .$$

Restatement of the B-T theorem

The closed unit ball $\mathbb{B}$ can be partitioned as $\mathbb{B} = A \cup B$ such that $A \sim_{SE_3} \mathbb{B}$ and $B \sim_{SE_3} \mathbb{B}$.

Proposition $\sim_G$ is an equivalence relation on $\mathcal{P}X$.

Exercise: prove this. (Transitivity is the interesting part.)

There is a related pre-ordering on $\mathcal{P}X$. Define

$$A \preccurlyeq_G B \iff \exists C \subseteq B \text{ s.t. } A \sim_G C$$

Theorem (Banach–Schröder–Bernstein theorem)
If $A \preccurlyeq_G B$ and $B \preccurlyeq_G A$ then $A \sim_G B$.

Proof Similar to the proof of the Schröder–Bernstein theorem. $\square$

Step 2 The notion of <u>G-paradoxical set</u>

A subset $E \subseteq X$ is said to be <u>G-paradoxical</u> if there exist disjoint $A_1, \dots, A_m, B_1, \dots, B_n \subseteq E$ and $g_1, \dots, g_m, h_1, \dots, h_n \in G$ such that

$$g_1 A_1 \cup \dots \cup g_m A_m = E = h_1 B_1 \cup \dots \cup h_n B_n$$

(N.B. we do not require $A_1 \cup \dots \cup A_m \cup B_1 \cup \dots \cup B_n = E$ or that the unions in the last line of the def. are disjoint.)

<u>Proposition</u> E is G-paradoxical iff it can be partitioned as $E = A \cup B$ where $A \sim_G E \sim_G B$.

<u>Proof</u> $\Leftarrow$ is obvious.

$\Rightarrow$ Suppose E is G-paradoxical, with $A_1, \dots, A_m$ and $B_1, \dots, B_n$.

By shrinking $A_1, \dots, A_m, B_1, \dots, B_n$ as necessary we can achieve that the unions

$$g_1 A_1 \cup \dots \cup g_m A_m = E = h_1 B_1 \cup \dots \cup h_n B_n$$

are disjoint.

So $E \sim_G A \subseteq E - B \subseteq E$ where $A = A_1 \cup \dots \cup A_m$, $B = B_1 \cup \dots \cup B_n$

Therefore $E \preccurlyeq_G E - B$ and $E - B \preccurlyeq_G E$ whence $E \sim_G E - B$ by the BSB theorem.

So $A' := E - B$ and $B' := B$ give $A' \sim_G E \sim_G B'$ and $A' \cup B' = E$ as required $\square$

By the proposition we have:

<u>Second restatement of B–T theorem</u> $\mathbb{B}$ is SE_3-paradoxical.

It is the second restatement of the B–T theorem we prove. We do this in a sequence of steps which we combine using:

<u>Lemma</u> If $E \subseteq X$ is G-paradoxical and $E' \sim_G E$ then E' is G-paradoxical.

Proof – exercise.

<u>Step 3</u> Paradoxicality of the free group F_2

F_2 is the free (non-abelian) group on 2 generators

Every element of F_2 can be written uniquely as a word over the alphabet $\{a, b, a^{-1}, b^{-1}\}$ that is reduced:

- The symbols a and a^{-1} are never adjacent to each other
- b b^{-1}

The identity element is the <u>empty word</u> ε

Group multiplication $w \cdot w'$ is by concatenation ww' followed by reduction.

Proposition F_2 is F_2-paradoxical

(F_2 acts on itself as left action.)

Proof $F_2 = \{\varepsilon\} \cup \underbrace{W(a)}_{A_1} \cup \underbrace{W(a^{-1})}_{A_2} \cup \underbrace{W(b)}_{B_1} \cup \underbrace{W(b^{-1})}_{B_2}$ disjoint union

all reduced words starting with a

We have $F_2 = W(a) \cup aW(a^{-1}) = A_1 \cup aA_2$
$= W(b) \cup bW(b^{-1}) = B_1 \cup bB_2$ $\square$

Step 4 Paradoxicality of the 2-sphere $\mathbb{S}^2$

$\left(\mathbb{S}^2 := \{(x,y,z) \mid x^2+y^2+z^2 = 1\} \right)$

Lemma There is a group embedding $F_2 \to SO_3$

(SO_3 is the group of 3-dimensional rotations around $\underline{0}$.)

Proof Fiddly!

Let $G \leq SO_3$ be the subroup given by the image of the embedding $F_2 \to SO_3$. N.B. G is countable!

We have an action of G on $\mathbb{S}^2$ (inherited from the action of SO_3).

this is a rotation

For every $g \in G$, the action $\underline{x} \mapsto g.\underline{x}$ has 2 fixed points.

Let $D \subseteq \mathbb{S}^2$ be the countable set of all these fixed points.

G also acts on $\mathbb{S}^2 - D$.

Let K be a subset of $\mathbb{S}^2 - D$ containing exactly one element from each orbit of G. Here we use AC!

__Lemma__ $\mathbb{S}^2 - D$ is G-paradoxical, hence SO_3-paradoxical.

__Proof__ $\mathbb{S}^2 - D = K \cup \underbrace{W(a)K}_{A_1} \cup \underbrace{W(a^{-1})K}_{A_2} \cup \underbrace{W(b)K}_{B_1} \cup \underbrace{W(b^{-1})K}_{B_2}$

(The union is disjoint because we excluded fixed points.)

We have
$$\mathbb{S}^2 - D = W(a)K \cup aW(a^{-1})K$$
$$= W(b)K \cup bW(b^{-1})K$$
$\square$

__Lemma__ For any countable subset $D \subseteq \mathbb{S}^2$ we have $\mathbb{S}^2 \sim_{SO_3} \mathbb{S}^2 - D$.

__Proof__ Find a rotation ρ such that $D, \rho D, \rho^2 D, \dots$ are all disjoint.

Then $\mathbb{S}^2 = D^* \cup (\mathbb{S}^2 - D^*) \sim_{SO_3} \rho D^* \cup (\mathbb{S}^2 - D^*) = \mathbb{S}^2 - D$

where $D^* = D \cup \rho D \cup \rho^2 D \cup \cdots$

$\square$

<u>Corollary</u> $\mathbb{S}^2$ is SO_3-paradoxical

<u>Step 5</u> Paradoxicality of $\mathbb{B}$ – the B-T Theorem!

<u>Lemma</u> $\mathbb{B} - \{\underline{0}\}$ is SO_3-paradoxical, hence SE_3-paradoxical.

<u>Proof</u> Follows straightforwardly from the SO_3-paradoxicality of $\mathbb{S}^2$.

<u>Lemma</u> $\mathbb{B} - \{\underline{0}\} \sim_{SE_3} \mathbb{B}$.

<u>Proof</u> Let ρ be an irrational rotation around the axis $(\frac{1}{2}, y, 0)$
(N.B. ρ is an element of SE_3 but not of SO_3.)

Then $\underline{0}, \rho\underline{0}, \rho^2\underline{0}, \rho^3\underline{0}, \dots$ are all distinct elements of $\mathbb{B}$

Define $Z^* := \{\rho^n(\underline{0}) \mid n \in \mathbb{N}\}$

We have

$$\mathbb{B} = Z^* \cup (\mathbb{B} - Z^*) \sim_{SE_3} \rho(Z^*) \cup (\mathbb{B} - Z^*) = \mathbb{B} - \{\underline{0}\} \quad \square.$$

<u>Corollary</u> $\mathbb{B}$ is SE_3-paradoxical.

That is the B-T Theorem holds!