Cardinal arithmetic with choice

$\boxed{\text{Assume AC}}$

So every set is well-orderable.

Consequences for the notion of cardinality:

- The cardinality of every infinite set is an aleph

  For every infinite X there exists α s.t. $|X| = \aleph_\alpha$

- The cardinality of every infinite set is its own square

  For every infinite X $\qquad |X| = |X| \cdot |X|$

- Cardinalities form a total order

  $|X| \leq |Y|$ or $|Y| \leq |X|$

- Very strict inequality coincides with strict inequality

  $|X| \ll |Y| \iff |X| < |Y|$ (exercise)

Lecture 7 now characterises the operations of $+$ and $\cdot$ on infinite cardinalities.

Infinite sums and products of cardinalities

Recall the definitions of indexed sum & product of sets.
Given $X_{(\cdot)} : I \to \underline{S}$ (the class of all sets)

$$\sum_{i\in I} X_i := \{(i,x) \mid i \in I,\ x \in X_i\}$$

$$\prod_{i\in I} X_i := \{x_{(-)} \mid x_{(-)} \text{ is a function with domain } I \text{ and } \forall i \in I\ x_i \in X_i\}$$

Given $(X_i)_{i\in I}$, $(Y_i)_{i\in I}$ and $(f_i : X_i \to Y_i)_{i\in I}$ define

$$\sum_{i\in I} f_i : \sum_{i\in I} X_i \to \sum_{i\in I} Y_i$$

$$(i,x) \longmapsto (i, f_i(x))$$

$$\prod_{i\in I} f_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$$

$$x_{(-)} \longmapsto (i \mapsto f_i(x_i))$$

<u>Proposition</u> The above operations are functorial. (They preserve identity functions and composition.)

Proposition (AC!) Given $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ such that $|X_i| \leq |Y_i|$ for all $i\in I$, it holds that:

$$\left|\sum_{i\in I} X_i\right| \leq \left|\sum_{i\in I} Y_i\right|$$

$$\left|\prod_{i\in I} X_i\right| \leq \left|\prod_{i\in I} Y_i\right|$$

Proof For every $i\in I$, $|X_i| \leq |Y_i|$ so there exists an injection $f_i : X_i \to Y_i$. $\boxed{\text{By AC}}$ we have $(f_i)_{i\in I}$. So we obtain functions

$$\sum_{i\in I} f_i : \sum_{i\in I} X_i \to \sum_{i\in I} Y_i \quad \text{and} \quad \prod_{i\in I} f_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i .$$

These are both injections because the component f_i are. $\boxed{\text{Exercise.}}$

$\square$

Corollary (AC!) If $|X_i| = |Y_i|$ for all $i\in I$, then

$$\left|\sum_{i\in I} X_i\right| = \left|\sum_{i\in I} Y_i\right| \quad \text{and} \quad \left|\prod_{i\in I} X_i\right| = \left|\prod_{i\in I} Y_i\right|$$

By the above corollary, we have well-defined notions of indexed sum and product of cardinalities.

Namely if $(\kappa_i)_{i\in I}$ is a family of cardinalities then we have well-defined cardinalities

$$\sum_{i\in I} \kappa_i := \left| \sum_{i\in I} \downarrow \kappa_i \right|$$

the set of ordinals $< \kappa_i$

$$\prod_{i\in I} \kappa_i := \left| \prod_{i\in I} \downarrow \kappa_i \right|$$

We now investigate the arithmetic of these operations.

Sums are very well-behaved and have expected properties

$$\sum_{\alpha<\lambda} \kappa = \lambda \cdot \kappa$$ $\boxed{\text{Exercise}}$ ⊛

(Here and henceforth, λ, κ are cardinalities which we identify with initial ordinals)

<u>Theorem</u> Suppose λ is an infinite cardinal and $(\kappa_\alpha)_{\alpha<\lambda}$ are nonzero cardinals, then

$$\sum_{\alpha<\lambda} \kappa_\alpha = \lambda \cdot \left(\sup_{\alpha<\lambda} \kappa_\alpha \right)$$

(N.B. $\sup_{\alpha<\lambda} \kappa_\alpha$ is itself a cardinal. $\boxed{\text{Exercise}}$)

Proof Define $\kappa = \sup_{\alpha<\lambda} \kappa_\alpha$.

Then $\sum_{\alpha<\lambda} \kappa_\alpha \leq \sum_{\alpha<\lambda} \kappa = \lambda \cdot \kappa$ by ⊗ above.

Conversely each $\kappa_\alpha \leq \sum_{\alpha<\lambda} \kappa_\alpha$, so $\sup_{\alpha<\lambda} \kappa_\alpha \leq \sum_{\alpha<\lambda} \kappa_\alpha$

Also $\lambda = \sum_{\alpha<\lambda} 1$ (by ⊗) $\leq \sum_{\alpha<\lambda} \kappa_\alpha$ (because κ_α nonzero)

So $\lambda \cdot \kappa \leq \left(\sum_{\alpha<\lambda} \kappa_\alpha\right) \cdot \left(\sum_{\alpha<\lambda} \kappa_\alpha\right) = \sum_{\alpha<\lambda} \kappa_\alpha$

(because this is its own square). $\square$

Corollary If $\lambda \leq \sup_{\alpha<\lambda} \kappa_\alpha$ then

$$\sum_{\alpha<\lambda} \kappa_\alpha = \sup_{\alpha<\lambda} \kappa_\alpha \, . \qquad (\lambda \text{ infinite})$$

Proof Since $\kappa \cdot \kappa' = \max(\kappa, \kappa')$ for infinite cardinalities.

Infinite products of cardinalities are <u>much</u> trickier than infinite sums!

The result below looks weak, but it is as good as one can get, <u>and</u> it is unusual in establishing a <u>strict</u> inequality.

<u>König's theorem</u> If $(\kappa_i)_{i \in I}$ and $(\lambda_i)_{i \in I}$ are families of cardinals with $\kappa_i < \lambda_i$ for all $i \in I$ then

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$$

<u>Proof of König's theorem</u>

First we show $\leq$ by exhibiting an injection.

An injection from $\sum_{i \in I} \downarrow \kappa_i$ to $\prod_{i \in I} \downarrow \lambda_i$ is:

$$(i, \alpha) \longmapsto \left(j \longmapsto \begin{cases} \alpha & \text{if } j = i \\ \kappa_j & \text{if } j \neq i \end{cases} \right)$$

(Important property to note: $\kappa_j \in \downarrow \lambda_j - \downarrow \kappa_j$)

For strictness $<$, suppose for contradiction that we have a bijection from $\sum_{i \in I} \downarrow \kappa_i$ to $\prod_{i \in I} \downarrow \lambda_i$.

This gives us a partition $\prod_{i \in I} \downarrow \lambda_i = \bigcup_{i \in I} X_i$

where each X_i is the image of the set $(i, \downarrow \kappa_i)$ under the bijection. Hence $|X_i| = \kappa_i$.

For each $i \in I$, the set $Y_i = \{x_i \mid x_{(-)} \in X_i\}$ is a subset of $\downarrow\lambda_i$ of cardinality $\leq \kappa_i$.

So there exists a (smallest) $\alpha_i \in \downarrow\lambda_i - Y_i$.

Then $$i \mapsto \alpha_i \in \prod_{i \in I} \downarrow\lambda_i$$

but $$i \mapsto \alpha_i \notin \bigcup_{i \in I} X_i . \quad \rightarrow\leftarrow$$

$\square$

A special case of König's theorem.

For any κ

$$\kappa = \sum_{\alpha<\kappa} 1 < \prod_{\alpha<\kappa} 2 = 2^{\kappa}$$

So König's theorem includes Cantor's theorem that $|X| < |\mathcal{P}X|$ as a special case.

Cardinal arithmetic with choice (continued)

In general cardinal exponentiation is a special case of product

$$\boxed{\kappa^\lambda = \prod_{\alpha<\lambda} \kappa}$$

$\boxed{\text{Exercise}}$

Cardinal exponentiation also enjoys the expected relationship with product and sum

$$\left(\prod_{i\in I} \kappa_i\right)^\lambda = \prod_{i\in I} \left(\kappa_i^{\ \lambda}\right)$$

$$\kappa^{\left(\sum_{i\in I} \lambda_i\right)} = \prod_{i\in I} \left(\kappa^{\lambda_i}\right)$$

$\boxed{\text{Exercise}}$

There is no simple rule for calculating cardinal exponentiation in aleph notation.

The question

$$\boxed{\text{Find } \gamma \text{ such that } \aleph_\alpha^{\aleph_\beta} = \aleph_\gamma}$$

has no simple answer.

In fact, there is little one can prove about even where 2^κ lies in the aleph hierarchy.

Essentially, we can prove only 3 things. (Highlighted below.)

By Cantor's theorem $\kappa < 2^\kappa$; i.e.

$$\boxed{2^{\aleph_\alpha} \geq \aleph_{\alpha+1}}$$

A

Cantor's generalised continuum hypothesis

asserts that

$$\boxed{\text{GCH} \qquad 2^{\aleph_\alpha} = \aleph_{\alpha+1} \quad \text{for all } \alpha}$$

GCH is independent of the axioms.

GCH very much simplifies cardinal arithmetic.
But it cannot be proved.

In addition to A **all** we can prove are

$$\boxed{\alpha \leq \beta \quad \text{implies} \quad 2^{\aleph_\alpha} \leq 2^{\aleph_\beta}}$$ B

and the following more interesting result.

Theorem For every α, $cf(2^{\aleph_\alpha}) > \aleph_\alpha$
(where $cf(\kappa)$ is the cofinality of κ as defined below) C

Cofinality

Let $(\alpha_\beta)_{\beta<\gamma}$ be a (transfinite) sequence of ordinals of length γ. The sequence is increasing if

$$\beta < \beta' < \gamma \quad \text{implies} \quad \alpha_\beta < \alpha_{\beta'}$$

If $(\alpha_\beta)_{\beta<\gamma}$ is increasing and γ is a limit ordinal then we define the limit of the sequence by

$$\lim_{\beta\to\gamma} \alpha_\beta \quad := \quad \sup_{\beta<\gamma} \alpha_\beta$$

N.B. the limit is a limit ordinal. $\boxed{\text{Exercise}}$

Def. (Cofinality) If α is a limit ordinal then its cofinality ($cf(\alpha)$) is the smallest ordinal γ for which α is the limit of an increasing sequence of length γ.

Examples

$$\mathrm{cf}(\aleph_0) = \mathrm{cf}(\omega) = \omega = \aleph_0$$

$$\omega = \lim 0, 1, 2, 3, 4, 5, \ldots$$

$$\mathrm{cf}(\omega^2) = \omega$$

$$\omega^2 = \lim \omega, \omega\cdot 2, \omega\cdot 3, \omega\cdot 4, \ldots$$

The cofinality of any countable limit ordinal is ω. Exercise.

$$\mathrm{cf}(\aleph_1) = \mathrm{cf}(\omega_1) = \omega_1 = \aleph_1$$

$$\omega_1 = \lim_{\alpha \to \omega_1} \alpha$$ but ω_1 is not a limit of any countable sequence of countable ordinals. Exercise

$$\mathrm{cf}(\aleph_2) = \mathrm{cf}(\omega_2) = \omega_2 = \aleph_2$$

$$\ldots$$

$$\mathrm{cf}(\aleph_\omega) = \mathrm{cf}(\omega_\omega) = \omega = \aleph_0$$

$$\omega_\omega = \lim \omega, \omega_1, \omega_2, \omega_3, \ldots$$

The cofinality of a cardinal κ, is simply $cf(\kappa)$ with κ considered as initial ordinal

For any ordinal α, $cf(\alpha)$ is an initial ordinal.

This will be proved in a future lecture.

Before proving the theorem we note some consequences

By theorem, $cf(2^{\aleph_0}) > \aleph_0$

Therefore we know at least that $2^{\aleph_0} \neq \aleph_\omega$

Similarly $cf(2^{\aleph_1}) > \aleph_1$ so $2^{\aleph_1} \neq \aleph_\omega$

etc.

Lemma Every infinite cardinal κ can be expressed as $\kappa = \sum_{\alpha < cf(\alpha)} \kappa_\alpha$ where $\kappa_\alpha < \kappa$ for every $\alpha < cf(\alpha)$

Proof We have $\kappa = \lim_{\alpha \to cf(\kappa)} \beta_\alpha$

where $(\beta_\alpha)_{\alpha < cf(\kappa)}$ is an increasing sequence.

So we can write $\downarrow\kappa$ as a disjoint union

$$\downarrow\kappa = \bigcup_{\alpha < cf(\kappa)} \left(\downarrow\beta_\alpha - \bigcup_{\alpha' < \alpha} \downarrow\beta_{\alpha'} \right) = \bigcup_{\alpha < cf(\kappa)} B_\alpha$$

Each B_α is a subset of $\downarrow\beta_\alpha$

Since $\beta_\alpha < \kappa$ and κ is an initial ordinal $|\downarrow\beta_\alpha| < \kappa$

So $|B_\alpha| < \kappa$.

Since $\bigcup_{\alpha < cf(\kappa)} B_\alpha$ is disjoint, we have

$$\downarrow \kappa = \bigcup_{\alpha < cf(\kappa)} B_\alpha \cong \sum_{\alpha < cf(\kappa)} B_\alpha \qquad \circledast$$

So $$\kappa = \sum_{\alpha < cf(\kappa)} \kappa_\alpha \quad \text{where } \kappa_\alpha = |B_\alpha| < \kappa.$$

$\square$

Exercise regarding $\circledast$.

If $(X_i)_{i \in I}$ is a family of disjoint sets

then $$\left| \bigcup_{i \in I} X_i \right| = \left| \sum_{i \in I} X_i \right| .$$

Proof of theorem

We need to show that $cf(2^{\aleph_\alpha}) > \aleph_\alpha$

By lemma

$$2^{\aleph_\alpha} = \sum_{\beta < cf(2^{\aleph_\alpha})} \kappa_\beta \quad \text{where every } \kappa_\beta < 2^{\aleph_\alpha}$$

By König's theorem

$$2^{\aleph_\alpha} = \sum_{\beta < cf(2^{\aleph_\alpha})} \kappa_\beta < \prod_{\beta < cf(2^{\aleph_\alpha})} 2^{\aleph_\alpha} = (2^{\aleph_\alpha})^{cf(2^{\aleph_\alpha})}$$

Suppose for contradiction $cf(2^{\aleph_\alpha}) \leq \aleph_\alpha$

$$2^{\aleph_\alpha} < (2^{\aleph_\alpha})^{cf(2^{\aleph_\alpha})} \leq (2^{\aleph_\alpha})^{\aleph_\alpha} \leq 2^{\aleph_\alpha \cdot \aleph_\alpha} = 2^{\aleph_\alpha}$$

$\rightarrow\leftarrow$

$\square$