(For this lecture we continue to assume AC.)

Singular, regular & inaccessible cardinals

An increasing sequence of ordinals of length γ (an ordinal) is a family $(\alpha_\beta)_{\beta<\gamma}$ of ordinals satisfying $\beta < \beta' < \gamma \Rightarrow \alpha_\beta < \alpha_{\beta'}$

If $(\alpha_\beta)_{\beta<\gamma}$ is an increasing sequence of ordinals of length γ where γ is a limit ordinal, then we define its limit by

$$\lim_{\beta \to \gamma} \alpha_\beta := \sup_{\beta<\gamma} \alpha_\beta$$

N.B. the limit is itself a limit ordinal.

The cofinality of a limit ordinal α (notation $\mathrm{cf}(\alpha)$) is the smallest limit ordinal γ for which α is the limit of an increasing sequence of ordinals of length γ

Cofinality examples (from last week)

$cf(\aleph_0) = cf(\omega) = \omega = \aleph_0$ Regular

$\omega = \lim 0, 1, 2, 3, 4, 5, \ldots$

$cf(\omega^2) = \omega = \aleph_0$

$\omega^2 = \lim \omega, \omega\cdot 2, \omega\cdot 3, \omega\cdot 4, \ldots$

The cofinality of any countable limit ordinal is $\omega = \aleph_0$

$cf(\aleph_1) = cf(\omega_1) = \omega_1 = \aleph_1$ Regular

$\omega_1 = \lim_{\alpha \to \omega_1} \alpha$ but ω_1 is not a limit of any countable sequence of countable ordinals.

$cf(\aleph_2) = cf(\omega_2) = \omega_2 = \aleph_2$ Regular

...

$cf(\aleph_\omega) = cf(\omega_\omega) = \omega = \aleph_0$ Singular

$\omega_\omega = \lim \omega, \omega_1, \omega_2, \omega_3, \ldots$

$cf(\aleph_{\omega+1}) = \aleph_{\omega+1}$ regular

$cf(\aleph_{\omega+2}) = \aleph_{\omega+2}$ regular

...

$cf(\aleph_{\omega\cdot 2}) = \aleph_0$ Singular

exercise

Lemma For any limit ordinal α

1. $cf(\alpha) \leq \alpha$
2. If α is not a cardinal (initial ordinal) then $cf(\alpha) < \alpha$
3. $cf(cf(\alpha)) = cf(\alpha)$.

Proof. $\boxed{\text{Exercise}}$

Proposition. If α is an infinite cardinal then $cf(\alpha)$ is also an infinite cardinal.

Proof Immediate from lemma. $\square$

An infinite cardinality κ is <u>regular</u> if $cf(\kappa) = \kappa$

κ is <u>singular</u> if $cf(\kappa) < \kappa$.

An infinite cardinality $\aleph_\alpha$ is said to be a:

<u>limit cardinality</u> if α is a limit ordinal

<u>successor cardinality</u> if α is a successor ordinal

Question A: Is every successor cardinality regular?

Question B: Is every uncountable limit cardinality singular?

We shall prove that the answer to question A is <u>yes</u>.

(AC is crucial for this to be the case.)

Recall from last week

Lemma A Every infinite cardinal κ can be expressed as

$$\kappa = \sum_{\alpha < cf(\kappa)} \kappa_\alpha \quad \text{where } \kappa_\alpha < \kappa \text{ for every } \alpha < cf(\kappa)$$

Lemma B A cardinal κ is singular iff we can express it as $\kappa = \sum_{i \in I} \kappa_i$ where $|I| < \kappa$ and $\kappa_i < \kappa$ for every $i \in I$.

Proof $\underline{\Rightarrow}$ Immediate from Lemma A because $cf(\kappa) < \kappa$.

$\underline{\Leftarrow}$ Suppose $\kappa = \sum_{i \in I} \kappa_i$ as in statement of lemma.

Then $\kappa = |I| \cdot \sup_{i \in I} \kappa_i = \max(|I|, \sup_{i \in I} \kappa_i)$

Since $|I| < \kappa$, this means $\kappa = \sup_{i \in I} \kappa_i$

By transfinitely ordering the κ_i elements without repetition we obtain $\kappa = \lim_{\alpha \to \lambda} \kappa_{i_\alpha}$ for some λ with $\lambda \leq |I| < \kappa$.

Hence $cf(\alpha) \leq \lambda < \kappa$. Hence κ is singular. $\square$

We can now answer Question A positively.

<u>Theorem</u> Every successor cardinal is regular.

<u>Proof</u> Suppose for contradiction that $\aleph_{\alpha+1}$ is singular.

By lemma B, $\aleph_{\alpha+1} = \sum_{i \in I} \kappa_i$ where $|I|, \kappa_i < \aleph_{\alpha+1}$

So $|I|, \kappa_i \leq \aleph_\alpha$ for all $i \in I$. Therefore:

$$\aleph_{\alpha+1} = \sum_{i \in I} \kappa_i \leq \sum_{i \in I} \aleph_\alpha = |I| \cdot \aleph_\alpha$$

$$\leq \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$$

That is $\aleph_{\alpha+1} \leq \aleph_\alpha$, which is absurd. $\rightarrow\leftarrow$

$\square$

Question B cannot be answered from the axioms we have (assuming the axioms are consistent).

A cardinal κ is said to be <u>weakly</u> <u>inaccessible</u> if it is an uncountable regular limit cardinal.

From the axioms we have, we cannot show that a weakly inaccessible cardinal exists.

We could add as a further axiom

(WIC) A weakly inaccessible cardinal exists

This is a modest example of a <u>large</u> <u>cardinal axiom</u>.

Singular, regular & inaccessible cardinals (continued)

Suppose we have a weakly inaccessible cardinal $\aleph_\lambda$

Of course $\aleph_\lambda = \lim_{\alpha \to \lambda} \aleph_\alpha$ as λ is a limit ordinal

Because $\aleph_\lambda$ is regular $\lambda \geqslant cf(\aleph_\lambda) = \aleph_\lambda$

But trivially $\lambda \leqslant \aleph_\lambda$

So: $\boxed{\lambda = \aleph_\lambda}$

Note that limit cardinals are exactly the cardinals $\aleph_\alpha$ that satisfy:

$$\boxed{\text{if } \aleph_\beta < \aleph_\alpha \text{ then } \aleph_{\beta+1} < \aleph_\alpha}$$

<u>Def.</u> A <u>strong limit cardinal</u> is a cardinal κ satisfying:

$$\boxed{\text{if } \kappa' < \kappa \text{ then } 2^{\kappa'} < \kappa}$$

<u>Proposition</u>

1. Every strong limit cardinal is a limit cardinal.
2. (GCH) Every limit cardinal is a strong limit cardinal.

If we do not assume the generalised continuum hypothesis.

Consider

$$\aleph_\omega = \lim_{n \to \omega} \aleph_0, \aleph_1, \aleph_2, \aleph_3 \cdots \aleph_n \cdots$$

This is clearly a limit cardinal

We <u>cannot</u> <u>prove</u> that it is a strong limit cardinal

Nonetheless we can demonstrate examples of strong limit cardinals.

Define $\beth_0 := \aleph_0$ "Beth$_0$"

$$\beth_{n+1} := 2^{\beth_n}$$

a strong limit carding

$$\beth_\omega := \lim_{n \to \omega} \beth_n = \lim \beth_0, \beth_1, \beth_2, \cdots$$

A <u>strongly inaccessible cardinal</u> is an uncountable regular strong limit cardinal.

<u>Proposition</u>

1. Every strongly inaccessible cardinal is weakly inaccessible.
2. (GCH) Every weakly inaccessible cardinal is strongly inaccessible.

We formulate (but do not assume) an axiom stating that strongly inaccessible cardinals exist

(SIC) There exists a strongly inaccessible cardinal

We know that (SIC) is not provable (if set theory is consistent)

The notion of a Grothendieck universe

This is used in various areas of mathematics (algebraic geometry, category theory, ...) in which one distinguishes between "small" and "large" sets. The idea is that all "small" sets can be collected inside a single "large" set – the Groth. universe.

A **Grothendieck universe** is a set $\mathcal{U}$ satisfying:

- It is transitive: for every set $X \in \mathcal{U}$ and element $x \in X$ we have $x \in \mathcal{U}$
- If $x, y \in \mathcal{U}$ then $\{x, y\} \in \mathcal{U}$
- For every set $X \in \mathcal{U}$, we have $\mathcal{P}X \in \mathcal{U}$.
- For every set $I \in \mathcal{U}$ and family $(X_i)_{i \in I}$, where every X_i is a set with $X_i \in \mathcal{U}$, we have $\left(\bigcup_{i \in I} X_i\right) \in \mathcal{U}$.

Given a Grothendieck universe $\mathcal{U}$, we call a set X <u>small</u> if $X \in \mathcal{U}$.

Because $\mathcal{U}$ is a set, we have a <u>set</u> of all small sets:

$$S := \{ X \in \mathcal{U} \mid X \text{ is a set} \}$$

One can prove that S (and $\mathcal{U}$) are not small sets.

In lecture 1 we axiomatised properties of the class of sets in the universe (class) $\underline{\mathcal{U}}$.

The axioms hold also for the (large) set S of small sets within the (large) Grothendieck universe $\mathcal{U}$.

Formally, $\mathcal{U}$ can be shown to be a <u>model</u> of our axioms in the sense of mathematical logic.

Theorem The following are equivalent.

1- A Grothendieck universe exists.

2- A strongly inaccessible cardinal exists.

Proof

(1) $\Rightarrow$ (2) Suppose U is a Groth. universe

Define

$$\kappa := \{\alpha \in U \mid \alpha \text{ is a von-Neumann ordinal}\}$$

One can prove that κ is itself a von-Neumann ordinal (but not small) that is a strongly inaccessible cardinal.

(2) $\Rightarrow$ (1) Suppose κ is a strongly inaccessible cardinal.

Then the set V_κ is a Grothendieck universe.

$\square$

see below

The cumulative hierarchy $\underline{V}$

By transfinite recursion we define sets $(V_\alpha)_{\alpha \in \underline{Ord}}$

$$V_0 := \emptyset$$

$$V_{\alpha+1} := \mathcal{P} V_\alpha$$

$$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$$

Define $\underline{V} := \bigcup_{\alpha} V_\alpha$

$\underline{V}$ is the proper class of all sets that are:

- hereditary sets

  (every element is a set and also a hereditary set)

- well-founded

  (the $\in$ relation is well-founded)

Brief glimpses at $(V_\alpha)_\alpha$

$\alpha \leq \beta \implies V_\alpha \subseteq V_\beta$

V_n has cardinality $\left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2^0}}}}}}\right\} n$ 2's

V_ω is the set of hereditarily finite well-founded sets

For any α, $\alpha \in V_{\alpha+1}$

In particular: $\omega \in V_{\omega+1}$

the von Neumann natural numbers

If $X \in V_\alpha$ then $\wp X \in V_{\alpha+1}$

In particular $\wp(\omega) \in V_{\omega+2}$

isomorphic to $\mathbb{R}$