Kardinalna Aritmetika: Izpit

June 15, 2018

This exam contains four questions. You are required to answer all four. All answers must be justified. You have 120 minutes to complete the exam. You may answer in English or in Slovene. Good luck!

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Seat (3.04)

Name and surname

Student ID

Question 1 (25 marks)

a) (6 marks) Given a class $\underline{X}$ and a class $\underline{Y}$ define what it means for a subclass $\underline{F} \subseteq \underline{X} \times \underline{Y}$ to be the graph of a surjective function from $\underline{X}$ to $\underline{Y}$.

b) (8 marks) Show that if $\underline{X}$ is a set then $\underline{F}$ is a set.

c) (8 marks) Show that if $\underline{X}$ and $\underline{Y}$ are both sets then the class of all surjective functions from $\underline{X}$ to $\underline{Y}$ is a set.

d) (3 marks) Suppose $\underline{X}$ is a set. Is the condition that $\underline{Y}$ be a set necessary for the class of all surjective functions from $\underline{X}$ to $\underline{Y}$ to be a set?

Question 2 (25 marks)

a) (5 marks) Give the equations that define ordinal exponentiation α^β by transfinite recursion.

b) (5 marks) Prove that for all ordinals α, $\alpha \leq \omega^\alpha$.

c) (6 marks) Define

$$\begin{aligned} \alpha_0 &:= \omega \\ \alpha_{n+1} &:= \omega^{\alpha_n}, \quad n < \omega \\ \gamma &:= \sup_{n<\omega} \alpha_n. \end{aligned}$$

Prove that $\gamma = \omega^{\gamma}$.

d) (6 marks) Suppose β is any ordinal such that $\beta = \omega^{\beta}$. Prove that $\gamma \leq \beta$.

e) (3 marks) Is it possible to find a cardinal κ such that $\kappa = \aleph_0{}^{\kappa}$ using cardinal exponentiation?

Question 3 (25 marks)

a) (8 marks) Prove that $|X|^{|Y|+|Z|} = |X|^{|Y|} \times |X|^{|Z|}$.

b) (12 marks) Let I be an arbitrary index set. Prove that for any cardinal κ and cardinals $\lambda_i \in I$,

$$\kappa^{\sum_{i\in I}\lambda_i} = \prod_{i\in I} \kappa^{\lambda_i}.$$

*(More space for your answer to part **(b)**.)*

c) (5 marks) Calculate $\prod_{i<\omega} 2^{\aleph_i}$.

Question 4 (25 marks)

In the space below and on the opposite page, answer **exactly one** of the two questions below.

1. Consider an $\mathbb{N}$-indexed sequence of surjective functions between sets

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \cdots$$

   Define the *inverse limit* of the sequence by

$$X_\infty := \{(x_i)_{i\in\mathbb{N}} \mid x_i = f_i(x_{i+1})\} .$$

   Show that every projection function $\pi_n : X_\infty \to X_n$ (defined by $\pi_n((x_i)_{i\in\mathbb{N}}) = x_n$) is surjective, and comment on whether or not your answer relies on any form of the Axiom of Choice.

2. Let $(P_\alpha)_{\alpha<2^{\aleph_0}}$ be a (transfinite) enumeration of all perfect subsets of $\mathbb{R}$. (It is a fact that $\mathbb{R}$ has $2^{\aleph_0}$-many perfect subsets.)

   (a) Define (transfinite) sequences $(x_\alpha)_{\alpha<2^{\aleph_0}}$ and $(y_\alpha)_{\alpha<2^{\aleph_0}}$ satisfying:

$$x_\alpha \neq y_\alpha \qquad \text{and} \qquad x_\alpha, y_\alpha \in P_\alpha \qquad \text{and} \qquad x_\alpha, y_\alpha \notin \bigcup_{\beta<\alpha} \{x_\beta, y_\beta\} .$$

   (b) Show that there exists an uncountable subset $X \subseteq \mathbb{R}$ that does not contain any perfect subset.

   (c) Comment on whether or not your answers to parts (a) and (b) rely on any form of the Axiom of Choice.

(Write your answer here.)

(More space for your answer to Question 4.)