Kardinalna Aritmetika: Izpit

June 29, 2018

This exam contains four questions. You are required to answer all four. All answers must be justified. You have 120 minutes to complete the exam. You may answer in English or in Slovene. Good luck!

Seat (3.04)

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Name and surname

Student ID

Question 1 (25 marks)

a) (6 marks) Let X be a set. Define what it means for $R \subseteq X \times X$ be an *equivalence relation*.

b) (6 marks) Let $R \subseteq X \times X$ be an equivalence relation. Recall that an *equivalence class* of R is a subset $Y \subseteq X$ for which there exists $y \in Y$ satisfying: for all $y' \in X$, it holds that $y' \in Y$ if and only if yRy'. Prove that the collection X/R of all equivalence classes is a set.

c) (7 marks) Suppose that X is well-orderable and R is an equivalence relation on X. Prove that $|X/R| \leq |X|$.

d) (6 marks) Suppose that X is well-orderable and that $|Y| \leq |X|$. Prove that Y is well-orderable.

Question 2 (25 marks)

a) (6 marks) If $(X, <_X)$ and $(Y, <_Y)$ are well-orders, define the well-order $(X, <_X) + (Y, <_Y)$.

b) (6 marks) Suppose $(X, <_X)$ and $(Z, <_Z)$ are well-orders and suppose $\mathrm{ord}(X, <_X) \leq \mathrm{ord}(Z, <_Z)$. Prove that there exists a well-order $(Y, <_Y)$ such that $\mathrm{ord}\big((X, <_X) + (Y, <_Y)\big) = \mathrm{ord}(Z, <_Z)$.

c) (8 marks) Suppose $\mathrm{ord}\big((X,<_X)+(Y,<_Y)\big) = \mathrm{ord}\big((X,<_X)+(Y',<_{Y'})\big)$. Prove that $\mathrm{ord}(Y,<_Y) = \mathrm{ord}(Y',<_{Y'})$.

d) (5 marks) Suppose $\mathrm{ord}\big((X,<_X)+(Y,<_Y)\big) = \mathrm{ord}\big((X',<_{X'})+(Y,<_Y)\big)$. Does it follow that $\mathrm{ord}(X,<_X) = \mathrm{ord}(X',<_{X'})$?

Question 3 (25 marks)

Recall the hierarchy of *beth* cardinals:

$$\begin{aligned} \beth_0 &:= \aleph_0 \\ \beth_{\alpha+1} &:= 2^{\beth_\alpha} \\ \beth_\lambda &:= \sup_{\alpha<\lambda} \beth_\alpha, \quad \lambda \text{ limit.} \end{aligned}$$

a) (5 marks) Is it true that $\beth_0 = (\beth_0)^{\aleph_0}$?

b) (8 marks) For $0 < n < \omega$ prove that $\beth_n = (\beth_n)^{\aleph_0}$.

*(More space for your answer to part **(b)**.)*

c) (4 marks) Recall the statement of Kőnig's theorem establishing a strict cardinal inequality.

d) (8 marks) Prove that $\beth_\omega < \left(\beth_\omega\right)^{\aleph_0}$.

Question 4 (25 marks)

In the space below and on the opposite page, answer **exactly one** of the two questions below.

1. (a) Suppose we have a function $f : \aleph_1 \to \aleph_0$. Prove that there exists $n < \aleph_0$ such that $\left|f^{-1}(n)\right| = \aleph_1$.

   (b) Suppose we have a function $f : \aleph_\omega \to \aleph_0$. Does there necessarily exist $n < \aleph_0$ such that $\left|f^{-1}(n)\right| = \aleph_\omega$?

   (c) Comment on whether or not your answers to parts (a) and (b) rely on any form of the Axiom of Choice.

2. This question concerns Borel sets of real numbers.

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. For any $n \in \mathbb{N}$, define $f^n : \mathbb{R} \to \mathbb{R}$ by:

   $$f^0(x) = x \qquad\qquad f^{n+1}(x) = f(f^n(x)) \,.$$

   (i) For any $\epsilon \in \mathbb{R}$ and $N \in \mathbb{N}$, show that $\left\{x \mid \forall m \geq N \ |f^m(x) - f^N(x)| \leq \epsilon\right\} \in \mathbf{\Pi^0_1}$.

   (ii) Show that $\left\{x \mid \text{the sequence } (f^n(x))_n \text{ is convergent}\right\} \in \mathbf{\Pi^0_3}$.

   (b) Recall from lectures that: (i) there are $2^{\aleph_0}$-many Borel sets; and (ii) every uncountable Borel set has a perfect subset. Using these facts, prove that there exists a family $\{B_\alpha\}_{\alpha<\omega_1}$ of Borel sets whose union $\bigcup_{\alpha<\omega_1} B_\alpha$ is *not* Borel. Hint: assume the Continuum Hypothesis (CH) and show that the statement follows; similarly, show that it follows from $\neg$CH.

(Write your answer here.)

(More space for your answer to Question 4.)