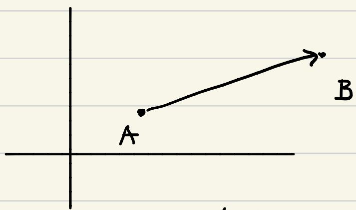
Seminar za
fizike

1. Vektorji v $\mathbb{R}^3$

Kaj je vektor v $\mathbb{R}^3$?



Vektor je usmerjena daljica med točkama A in B, $\overrightarrow{AB}$.

Vektorja sta kolinearna, če imata enako smer.
Vektorja sta ortogonalna, če oklepata pravi kot.
Velikost ali norma vektorja $\vec{v} = (a, b, c)$ je

$$v = \|\vec{v}\| = |\vec{v}| = \sqrt{a^2+b^2+c^2}.$$

Vektorjem, ki so veliki 1, pravimo enotski ali normirani.

Operacije z vektorji:

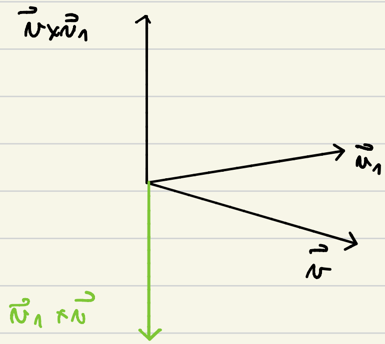
$+$: $(a, b, c) + (a_1, b_1, c_1) = (a+a_1, b+b_1, c+c_1)$

$\cdot$: $\lambda \cdot (a, b, c) = (\lambda a, \lambda b, \lambda c)$ (množenje s skalarjem)

$\langle\, ,\, \rangle$: $\langle (a, b, c), (a_1, b_1, c_1) \rangle = aa_1 + bb_1 + cc_1 =$

$= \|(a,b,c)\| \cdot \|(a_1, b_1, c_1)\| \cdot \cos\varphi$

$\times$: $\underset{\vec{v}}{\underset{\parallel}{(a, b, c)}} \times \underset{\vec{v}_1}{\underset{\parallel}{(a_1, b_1, c_1)}} =$



$$\vec{v}\times\vec{v}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix} =$$

$$= \vec{i}\cdot(b\cdot c_1) + \vec{j}\cdot c\cdot a_1 + \vec{k}\cdot a\cdot b_1 - (\vec{k}\cdot a_1\cdot b + \vec{j}\cdot a\cdot c_1 + \vec{i}\cdot b_1\cdot c)$$

$$= \vec{i}\cdot(bc_1 - b_1c) - \vec{j}\cdot(a\cdot c_1 - a_1\cdot c) + \vec{k}\,(a\cdot b_1 - a_1\cdot b)$$

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix} \end{array} = \left(\overset{a_2}{bc_1-b_1c},\ \overset{b_2}{ac_1-a_1c},\ \overset{c_2}{ab_1-a_1b}\right)$$

(razvoj po 1. vrstici)

<u>Primer</u>: Dana sta vektorja $\vec{v}=(1,2,1)$ in $\vec{v}_1=(1,0,-1)$.
Izračunaj vektorski produkt, skalarni produkt, vsoto in projekcijo $\vec{v}_1$ na $\vec{v}$.

$\vec{v}_1+\vec{v}=(2,2,0)$, $\langle\vec{v},\vec{v}_1\rangle = 1\cdot1+2\cdot0+1\cdot(-1)=0$

Vektorja $\vec{v}$ in $\vec{v}_1$ sta pravokotna, zato je projekcija $\vec{v}_1$ na $\vec{v}$ vektor 0.

$$\vec{v} \times \vec{v}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \vec{i}\,(2\cdot(-1)) + \vec{j}\cdot 1\cdot 1 + \vec{k}\cdot 1\cdot 0 - (1\cdot 2\cdot \vec{k} + 0\cdot 1\cdot \vec{i} + 1\cdot \vec{j}\,(-1)) =$$

$$= (-2, 2, -2)$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \left(2\cdot(-1) - 0\cdot 1, -(1\cdot(-1) - 1\cdot 1), 1\cdot 0 - 1\cdot 2\right)$$
$$= (-2, 2, -2)$$

$$\vec{k}: \quad \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \quad - \quad +$$

Preverimo rezultat: $\langle (-2, 2, -2), (1, 0, -1) \rangle = -2 + 0 + 2 = 0$
$\langle (-2, 2, -2), (1, 2, 1) \rangle = -2 + 4 - 2 = 0$

velikosti: $\|(1, 2, 1)\| = \sqrt{1+4+1} = \sqrt{6}$
$\|(1, 0, -1)\| = \sqrt{1+0+1} = \sqrt{2}$
$\|(-2, 2, -2)\| = \sqrt{4+4+4} = \sqrt{12} = \sqrt{2}\cdot\sqrt{6}$

Kako pa preverimo pravilo desnega vijaka?

Mešani produkt: $(\vec{v}, \vec{v}_1, \vec{v}_2) := \langle \vec{v} \times \vec{v}_1, \vec{v}_2 \rangle =$

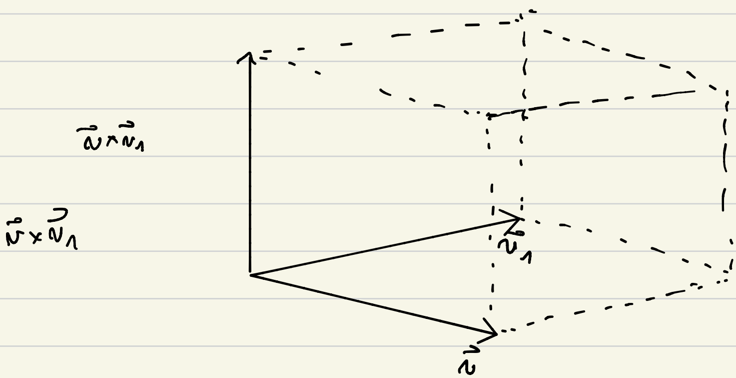
$$= \begin{vmatrix} a_2 & b_2 & c_2 \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$(\vec{v}, \vec{v}_1, \underbrace{\vec{v}\times\vec{v}_1}_{\vec{v}_2}) = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -2 & 2 & -2 \end{vmatrix} =$$

$$= 1\cdot\big(0\cdot(-2) - 2\cdot(-1)\big) - 2\cdot\big(1\cdot(-2) - (-2)(-1)\big) +$$

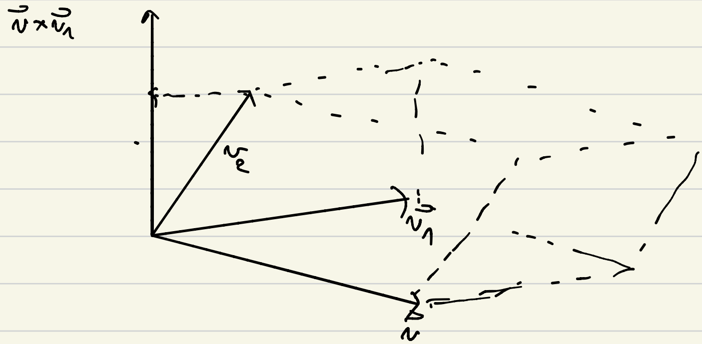
$$+ 1\cdot\big(1\cdot 2 - (-2)\cdot 0\big) = 1\cdot 2 - 2\cdot(-4) + 2 = 12$$

Interpretacija: če je ta determinanta pozitivna, so vektorji orientirani pozitivno oz. v smislu desnega vijaka. Ta količina predstavlja prostornino paralelepipeda, napetega na vektorje $\vec{v}, \vec{v}_1, \vec{v}_2$ (modulo predznak).



$$V = P_{osn}\cdot \text{višina} =$$
$$= \sqrt{12}\cdot\sqrt{12}$$

$V = P_{OP} \cdot \text{višina}$



Lastnosti skalarnega produkta:

$$\lambda \langle \vec{v}, \vec{v}_1 \rangle = \langle \lambda \vec{v}, \vec{v}_1 \rangle = \langle \vec{v}, \lambda \vec{v}_1 \rangle$$

$$\langle \vec{v} + \vec{v}_2, \vec{v}_1 \rangle = (a + a_2) \cdot a_1 + (b + b_2) \cdot b_1 + (c + c_2) \cdot c_1 =$$
$$= a a_1 + a_2 a_1 + b b_1 + b_2 b_1 + c c_1 + c_2 c_1 =$$
$$= \langle \vec{v}, \vec{v}_1 \rangle + \langle \vec{v}_2, \vec{v}_1 \rangle$$

$$\langle \vec{v}, \vec{v} \rangle = a^2 + b^2 + c^2 = \|\vec{v}\|^2$$

$$\langle \vec{v}, \vec{v}_1 \rangle = \langle \vec{v}_1, \vec{v} \rangle \qquad \text{(komutativnost)}$$

Lastnosti vektorskega produkta:

$$\lambda (\vec{v} \times \vec{v}_1) = (\lambda \vec{v}) \times \vec{v}_1 = \vec{v} \times (\lambda \vec{v}_1)$$
$$(\vec{v} + \vec{v}_2) \times \vec{v}_1 = \vec{v} \times \vec{v}_1 + \vec{v}_2 \times \vec{v}_1$$
$$\vec{v} \times \vec{v}_1 = - \vec{v}_1 \times \vec{v} \qquad \text{(antikomutativnost)}$$

Opazke: vektorski produkt je enak 0, ko sta vektorja kolinearna (0 je kolinearen z vsakim vektorjem).

Lastnosti mešanega produkta:

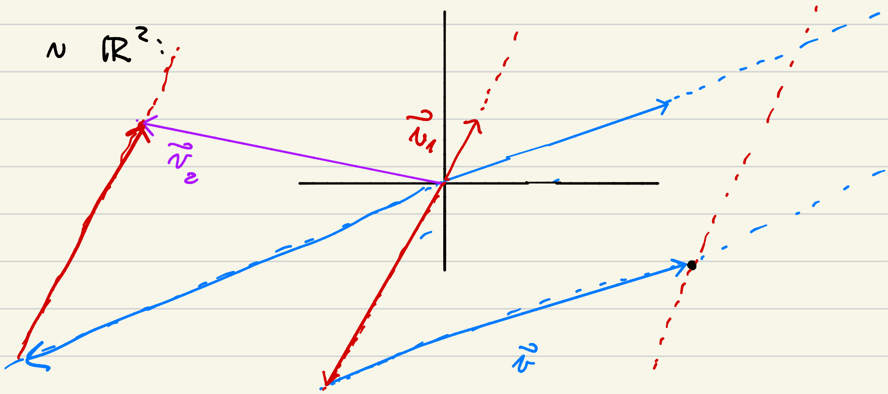
$$(\vec{v},\vec{v}_1,\vec{v}_2) = (\vec{v}_2,\vec{v},\vec{v}_1) = (\vec{v}_1,\vec{v}_2,\vec{v})$$
$$= -(\vec{v}_1,\vec{v},\vec{v}_2)$$

$$(\vec{v}+\vec{w},\vec{v}_1,\vec{v}_2) = (\vec{v},\vec{v}_1,\vec{v}_2)+(\vec{w},\vec{v}_1,\vec{v}_2) \quad \text{(distributivnost)}$$

Opazke: Mešani produkt je enak 0, ko so vektorji komplanarni.
Če je mešani produkt $(\vec{v},\vec{v}_1,\vec{v}_2)>0$, potem vektor $\vec{v}_2$ leži na isti strani ravnine, napete na $\vec{v}$ in $\vec{v}_1$, kot $\vec{v}\times\vec{v}_1$.

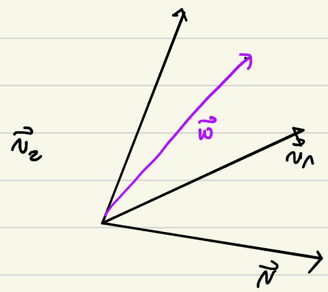
Linearna neodvisnost:

v $\mathbb{R}^2$:



Test: $\vec{v}$ in $\vec{v}_1$ sta lin. neodvisna, če je $\vec{v}\times\vec{v}_1 \neq 0$.
Ali je $\vec{v}_2$ lin. neodv. od $\vec{v}_1$ in $\vec{v}$? Ne. V ravnini so trije vektorji vedno lin. odvisni: Lahko najdemo števili λ in λ_1, da je $\vec{v}_2 = \lambda\cdot\vec{v} + \lambda_1\cdot\vec{v}_1$.

$\mathbb{R}^3$: lin. neodvisni so vsaki trije nekomplan. vektorji. Test: mešani produkt je neničeln.



Ali je $\vec{w}$ lin. odvisen od $\vec{v}, \vec{v}_1, \vec{v}_2$?

4 vektorji v $\mathbb{R}^3$ so vedno lin. odvisni.

Def: Vektorji $\vec{v}_1, \dots, \vec{v}_n$ so linearno neodvisni, če velja sklep:

$$\underbrace{\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 + \dots + \lambda_n\vec{v}_n = 0}_{} \iff \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

linearna kombinacija je trivialna natanko tedaj, ko so vsi koeficienti enaki 0.

Primeri: $\vec{v}_1 = (1,2)$, $\vec{v}_2 = (-2,1)$, $\vec{v}_3 = (1,1)$

$$\lambda_1 \cdot \vec{v}_1 + \lambda_2 \cdot \vec{v}_2 + \lambda_3 \cdot \vec{v}_3 = 0 \overset{?}{\Longrightarrow} \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$$

$$\parallel$$

$$(\lambda_1 - 2\lambda_2 + \lambda_3, 2\lambda_1 + \lambda_2 + \lambda_3) = 0$$

$$\left.\begin{array}{l} \lambda_1 - 2\lambda_2 + \lambda_3 = 0 \quad \cdot 2 \\ 2\lambda_1 + \lambda_2 + \lambda_3 = 0 \end{array}\right\} -$$

$$\left.\begin{array}{l} \\ \end{array}\right| \cdot 2 \Big\} +$$

$$-5\lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_3 = 5\lambda_2$$

ne sledi, da so vsi $\lambda_i = 0$.

$$5\lambda_1 + 3\lambda_3 = 0 \Rightarrow \lambda_3 = -\frac{5}{3}\lambda_1$$