VAJE

MATEMATIKA 2

Študijsko leto: 2020/2021

POGLAVJA:

- FUNKCIJE VEČ SPREMENLJIVK

Določi definicijsko območje in zalogo vrednosti funkcij f podanih z naslednjimi predpisi. Nariši definicijsko območje in nekaj nivojnic.

a) $f(x,y) = x^2 - y$

b) $f(x,y) = \sqrt{y^2 - 4x + 8}$

c) $f(x,y) = \ln(x + \sqrt{y-2})$

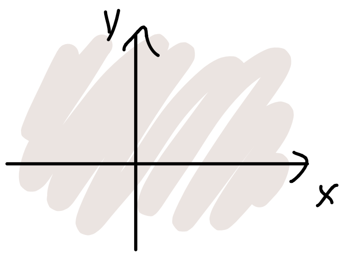
Realne funkcije dveh spremenljivk: $f : \mathbb{R}^2 \to \mathbb{R}$

$D_f \subseteq \mathbb{R}^2$

$Z_f \subseteq \mathbb{R}$

a) $f(x,y) = x^2 - y$

- $D_f = \mathbb{R}^2$



- $Z_f = \mathbb{R}$

$z \in \mathbb{R}$

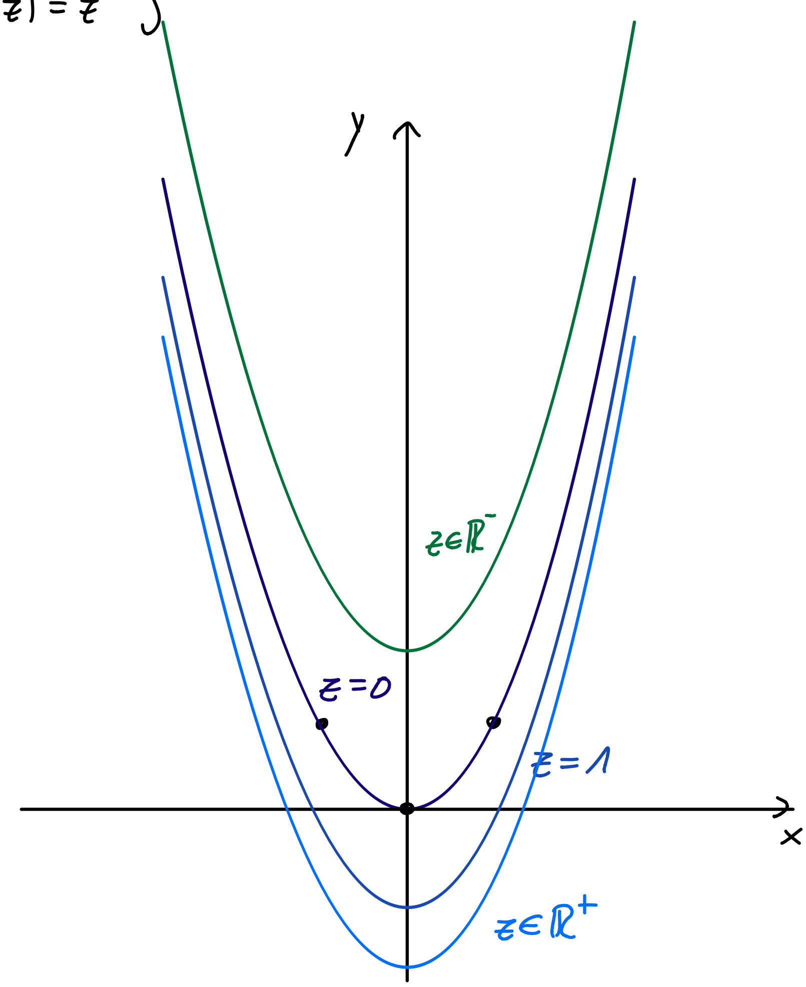
$$\left.\begin{aligned} f(x,y) &= z \\ f(0,y) &= -y \\ f(0,-z) &= z \end{aligned}\right\} \text{ pri } x=0 \text{ in } y=-z$$

- Nivojnice:

$z=0$: $\quad 0 = f(x,y)$
$0 = x^2 - y$
$y = x^2$

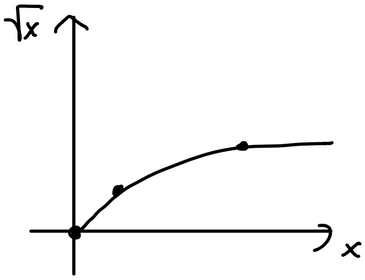
$z=1$: $\quad 1 = f(x,y)$
$y = x^2 - 1$

$z \in Z_f = \mathbb{R}$: $\quad z = x^2 - y$
$y = x^2 - z$



b) $f(x,y)=\sqrt{y^2-4x+8}$

$\sqrt{\ }:[0,\infty)\to[0,\infty)$

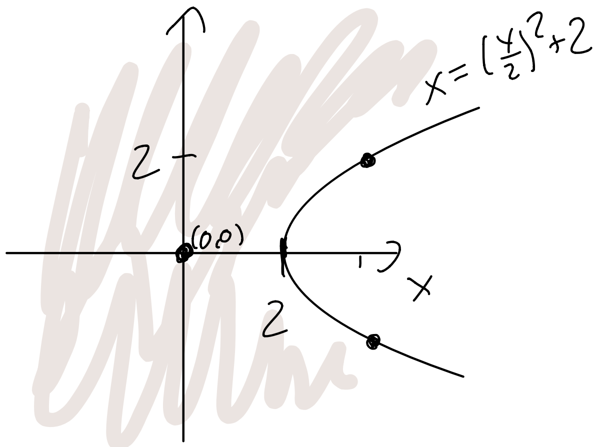


- $D_f \subseteq \mathbb{R}^2$

$y^2-4x+8 \geq 0$

$\vdots$

$y \geq 4x-8$ ali $x \leq \left(\frac{y}{2}\right)^2+2$



$D_f=\{(x,y)\in\mathbb{R}^2;\ x\leq\left(\frac{y}{2}\right)^2+2\}$

- $Z_f \subseteq \mathbb{R}_0^+=[0,\infty)$ (ker $\sqrt{\ }:\mathbb{R}_0^+\to\mathbb{R}_0^+$)

$Z_f=[0,\infty)$

$z\in\mathbb{R}_0^+$

$f(x,y)=z$

$\sqrt{z^2}=z$ (ker $z\in\mathbb{R}_0^+$)

$z^2=y^2-4x+8$

$z^2=y^2$ $\swarrow x=2$

$y=z$

$f(2,z)=z$ ✓

• Nivojnice:

$z=0$: $\quad 0=\sqrt{y^2-4x+8}$

$\vdots$

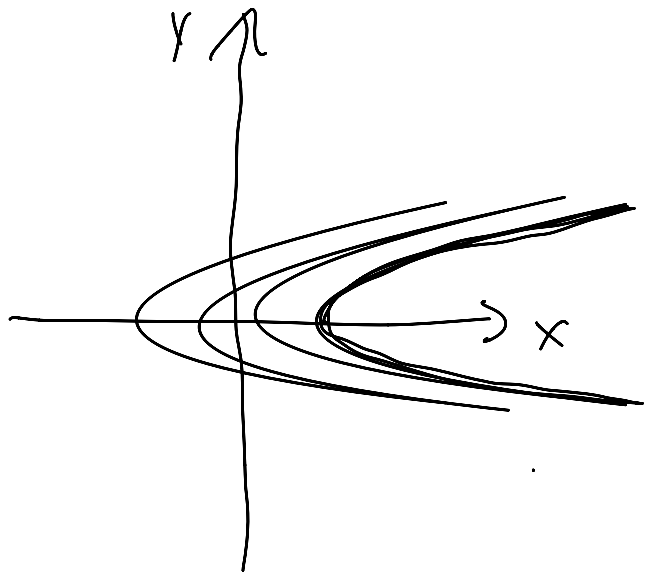
$x=\left(\frac{y}{2}\right)^2+2$

$z\in\mathbb{R}_0^+$ $\quad z=\sqrt{y^2-4x+8}$

$z^2=y^2-4x+8$

$\vdots$

$x=\left(\frac{y}{2}\right)^2+2-\left(\frac{z}{2}\right)^2$



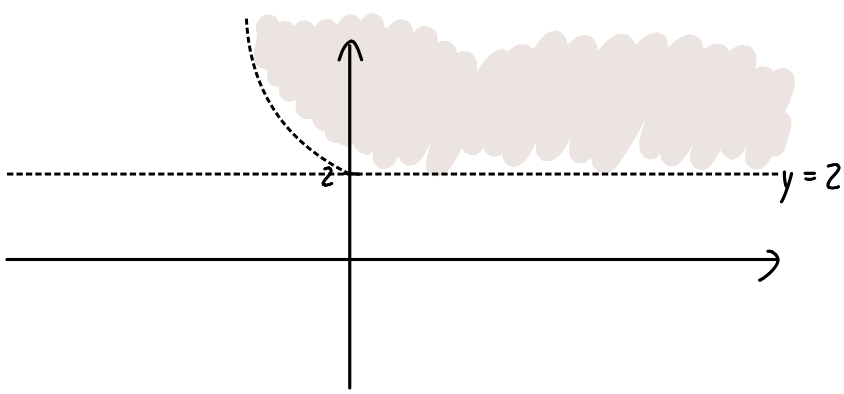
c) $f(x,y)=\ln(x+\sqrt{y-2})$

• $D_f\subseteq\mathbb{R}^2$

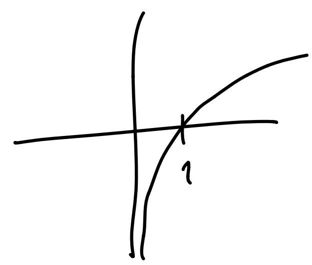
$x+\sqrt{y-2}>0$ zaradi $\ln$

$y\geq 2$ zaradi $\sqrt{\ }$

$D_f=\{(x,y)\in\mathbb{R}^2;\ x>-\sqrt{y-2},\ y\geq 2\}$



ln:



$\ln: (0,\infty)\to\mathbb{R}$

- $Z_f \subseteq Z_n = \mathbb{R}$

  $\underline{Z_f = \mathbb{R}}$

  $z \in \mathbb{R}$

  $z = f(x,y)$

  $z = \ln_e(x+\sqrt{y-2})$

  $y=2,\quad x=e^z$ ✓

- Nivojnice:

  $z = f(x,y)$

  $z = \ln_e(x+\sqrt{y-2})$

  $e^z = x+\sqrt{y-2}$

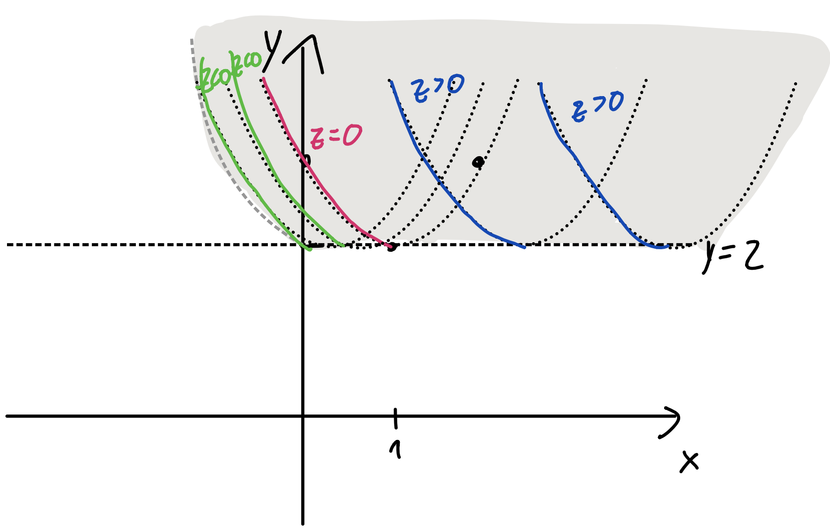
  $\sqrt{y-2} = e^z - x \quad /^2$

  $e^z - x \geq 0$ oz. $x \leq e^z$ (zato samo "leva stran")

  $y-2 = (e^z-x)^2$

  premik v desno za pozitivno vrednost e^z

  $y = (e^z-x)^2+2 = (x-e^z)^2+2$



Opomba:

Z vsemi nivojnicami pokrijemo celotno definicijsko območje.

Določi in nariši definicijsko območje naslednjih funkcij:

a) $f(x,y) = \frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$

b) $f(x,y) = \frac{\ln(1-|x|-|y|)}{xy}$

a) $f(x,y) = \frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$

$\sqrt{\ }: \ 4x - y^2 \geq 0 \qquad y \leq 4x$

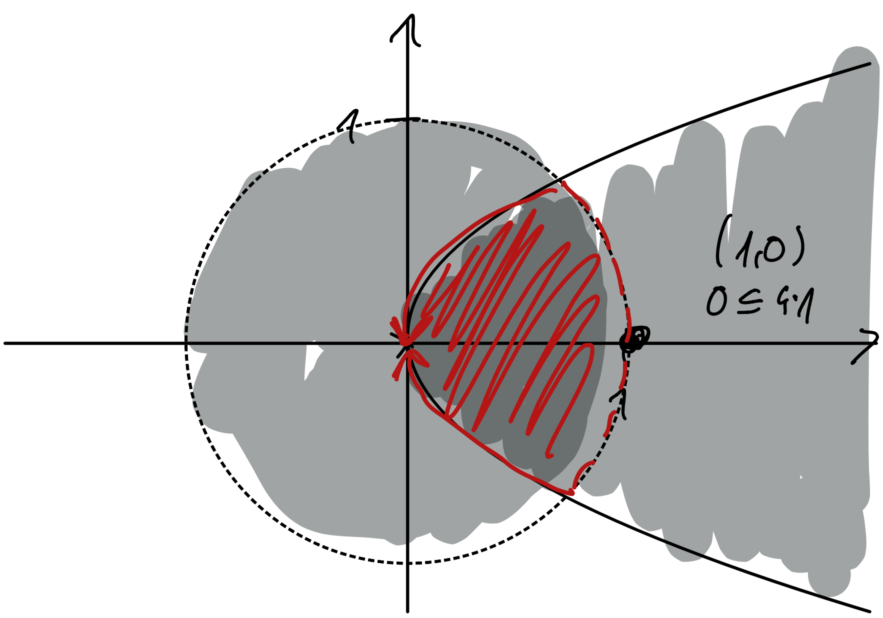
$\ln: \ 1 - x^2 - y^2 > 0 \qquad x^2 + y^2 < 1$

$\frac{1}{x}: \ \ln(1-x^2-y^2) \neq 0$

$1 - x^2 - y^2 \neq 1$

$\underset{\substack{\cup| \\ 0}}{x^2} + \underset{\substack{\cup| \\ 0}}{y^2} \neq 0 \qquad (x,y) \neq (0,0)$

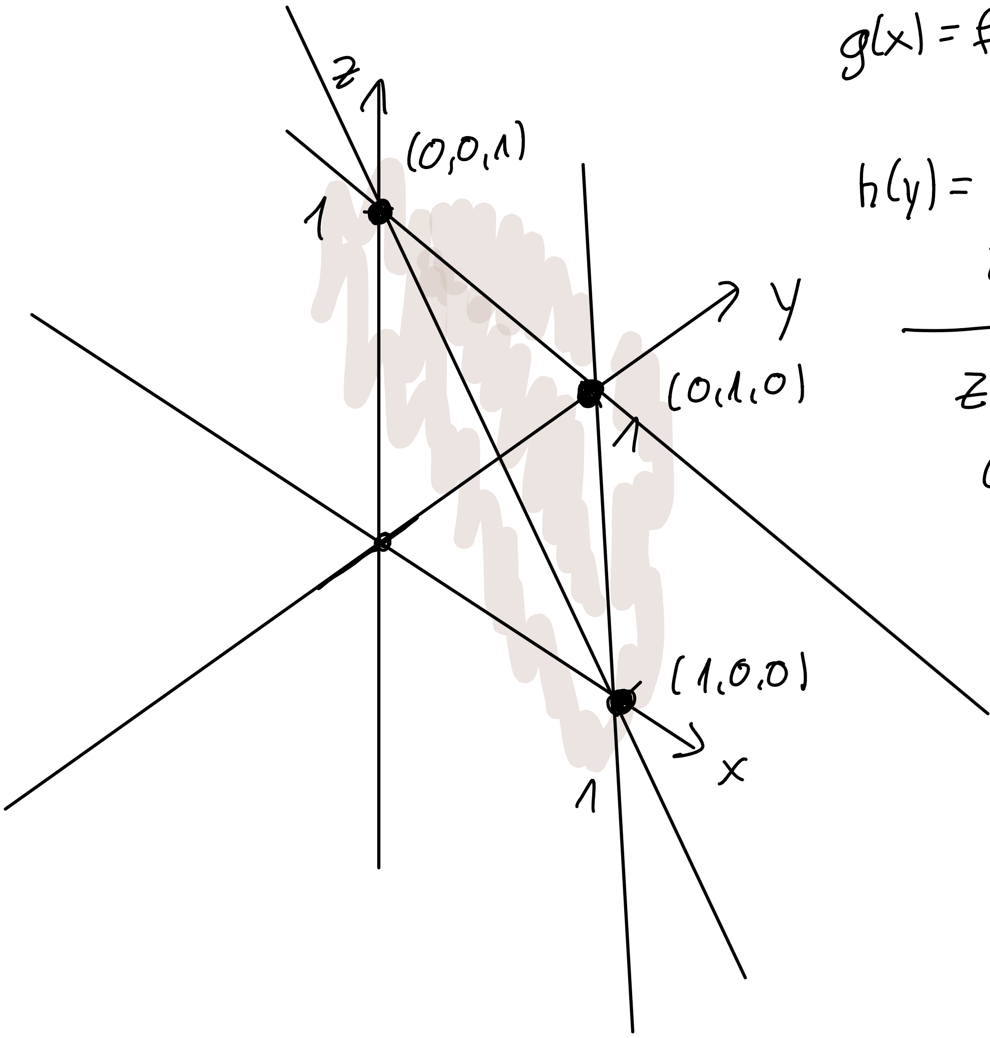
$D_f = \{ (x,y) \in \mathbb{R}^2 ; \ y^2 \leq 4x, \ x^2 + y^2 < 1, \ (x,y) \neq (0,0) \}$



Nariši grafe naslednjih funkcij:

a) $f(x,y) = 1 - x - y$ b) $f(x,y) = \sqrt{4 - x^2 - y^2}$

a) $f(x,y) = 1 - x - y$



$g(x) = f(x,0) = 1-x$

$z = 1-x$

$h(y) = f(0,y) = 1-y$

$z = 1-y$

$z = 0$

$0 = 1-x-y$

$y = 1-x$

b) $f(x,y) = \sqrt{4 - x^2 - y^2}$

- $D_f \subseteq \mathbb{R}^2$

  $4 - x^2 - y^2 \geq 0$

  $x^2 + y^2 \leq 2^2$

- $Z_f \subseteq \mathbb{R}$

  $$0 \leq 4 - \underbrace{x^2}_{\geq 0} - \underbrace{y^2}_{\geq 0} \leq 4$$

  $0 \leq \sqrt{4 - x^2 - y^2} \leq \sqrt{4}$

  $Z_f = [0, 2]$!

- Nivojnice:

  $z = 0$: $4 - x^2 - y^2 = 0$

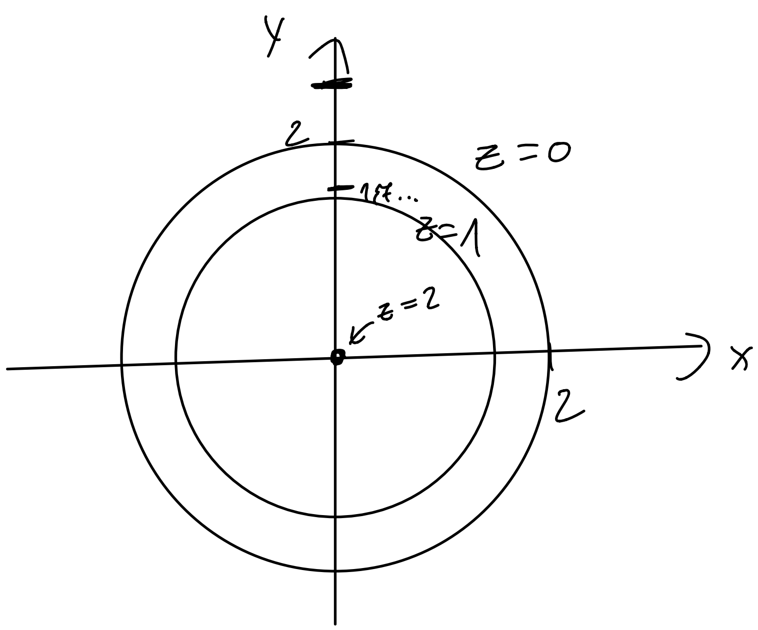
  $x^2 + y^2 = 2^2$

  $z = 1$: $\sqrt{4 - x^2 - y^2} = 1$

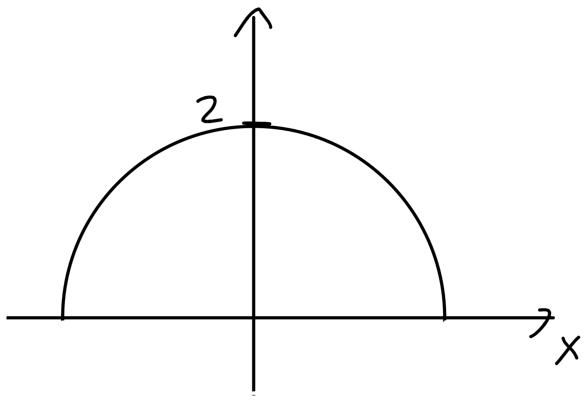
  $x^2 + y^2 = \sqrt{3}^2$

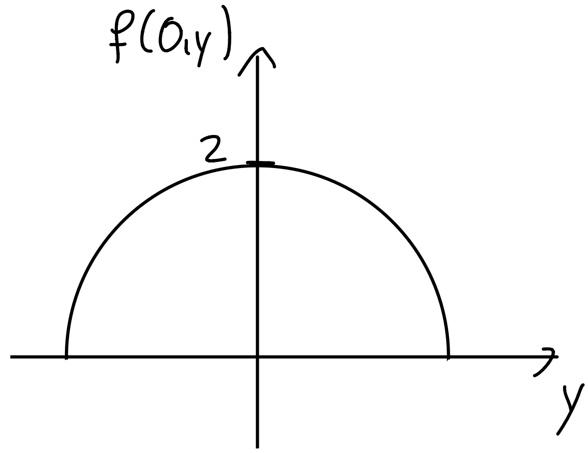
  $z = 2$: $\sqrt{4 - x^2 - y^2} = 2$

  $x^2 + y^2 = 0$

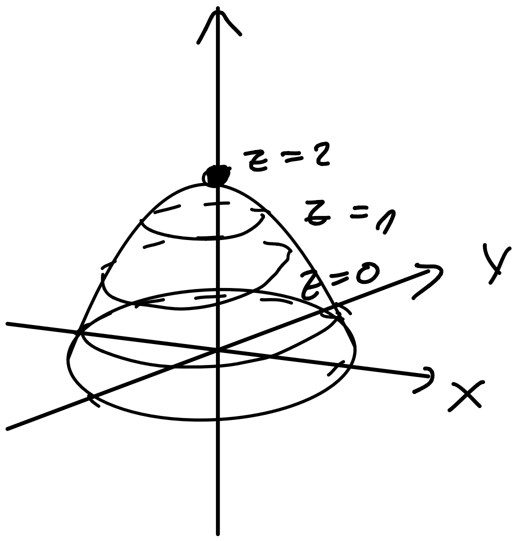


$f(x,0) = \sqrt{4-x^2}$





$f(x,y)$:



Določi preseke naslednjih funkcij:

a) $f(x,y) = \sqrt{2-x^2-y^2}$ in $g(x,y) = 1$

b) $f(x,y) = \sqrt{x^2+y^2}$ in $g(x,y) = x$

c) $f(x,y) = \sqrt{x^2+y^2}$ in $g(x,y) = x+1$

d) $f(x,y) = x^2+y^2$ in $g(x,y) = x$

a) $f(x,y) = \sqrt{2-x^2-y^2}$ in $g(x,y) = 1$

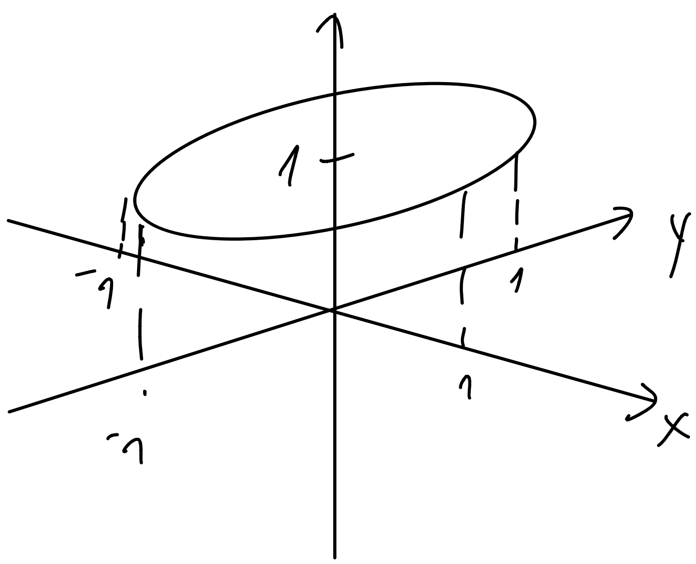
$f(x,y) = g(x,y)$

$\sqrt{2-x^2-y^2} = 1 \quad /^2$ $\qquad D_f: \; 2-x^2-y^2 \geq 0$

$2-x^2-y^2 = 1$ $\qquad x^2+y^2 \leq 2$

$x^2+y^2 = 1$ preverimo ✓

$z = 1$



b) $f(x,y) = \sqrt{x^2+y^2}$ in $g(x,y) = x$

$D_f : x^2+y^2 \geq 0$ ✓ vedno res

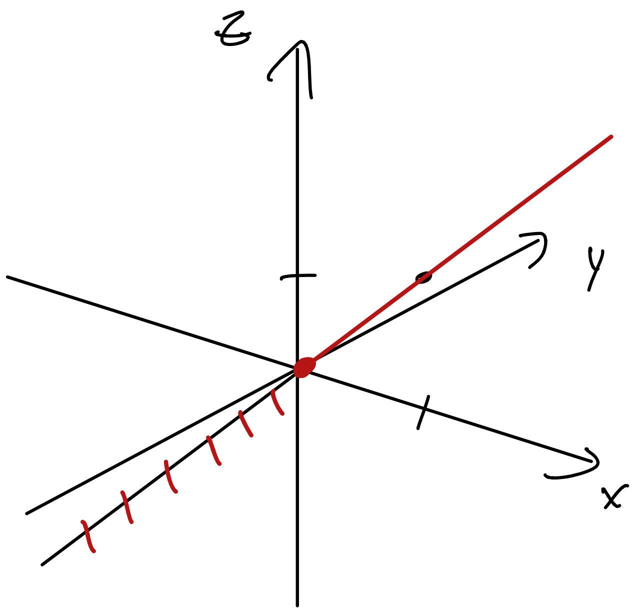
$f(x,y) = g(x,y)$

$\sqrt{x^2+y^2} = x \quad /^2 \qquad x \geq 0$ ker $\sqrt{} \geq 0$

$\cancel{x^2}+y^2 = \cancel{x^2}$

$y^2 = 0$

$y = 0, \ x \geq 0$



LIMITE IN POLARNE KOORDINATE

13.10.2020

$$\lim_{(x,y)\to(a,b)} f(x,y)=c \iff \lim_{r\downarrow 0}\ \sup_{0\le\varphi\le 2\pi} |f(a+r\cdot\cos\varphi,\ b+r\cdot\sin\varphi)-c|=0$$

* običajno: nove spremenljivke: $x=a+r\cdot\cos\varphi$
$y=b+r\cdot\sin\varphi$

Izračunaj naslednje limite.

(a) $\lim_{(x,y)\to(0,2)} \frac{\sin xy}{x}$

(b) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{(x^2+y^2)^2}$

(d) $\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2}$

(a) $\lim_{(x,y)\to(0,2)} \frac{\sin xy}{x}$

$\begin{pmatrix} x\to 0 \\ y\to 2 \end{pmatrix}$

$$= \lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \frac{2\cdot\sin x\cdot\cos x}{x} =$$

$$= \lim_{x\to 0} (2\cos x)\cdot\frac{\sin x}{x} =$$

$$= 2\left(\lim_{x\to 0}\cos x\right)\cdot\left(\lim_{x\to 0}\frac{\sin x}{x}\right) =$$

$$= 2\cdot 1\cdot 1 = \underline{\underline{2}}$$

(b) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

$x=r\cdot\cos\varphi$
$y=r\cdot\sin\varphi$

$$= \lim_{r\downarrow 0} \frac{r\cdot\cos\varphi\cdot r^2\cdot\sin^2\varphi}{r^2\cdot\cos^2\varphi+r^2\cdot\sin^2\varphi} =$$

$$= \lim_{r\downarrow 0} \frac{r^{\cancel{3}}\cdot\cos\varphi\cdot\sin^2\varphi}{\cancel{r^2}} =$$

$$= \underbrace{\cos\varphi\cdot\sin^2\varphi}_{\text{omejeno}}\cdot\lim_{r\downarrow 0} r =$$

$$= \underline{\underline{0}}$$

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{(x^2+y^2)^2}$ $= \lim_{r\downarrow 0} \frac{r^2\cos^2\varphi \cdot r^2 \sin^2\varphi}{(r^2)^2} =$

$x = r\cdot\cos\varphi$
$y = r\cdot\sin\varphi$

$= \lim_{r\downarrow 0} \cos^2\varphi \cdot \sin^2\varphi =$

$= \cos^2\varphi \cdot \sin^2\varphi$ (odvisno od kota φ)

$\Rightarrow$ LIMITA NE OBSTAJA

(d) $\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2}$ $= \lim_{r\downarrow 0} \frac{(r\cdot\cos\varphi)^2 \cdot \ln(1+r\cdot\cos\varphi)}{(r\cdot\cos\varphi)^2 + (r\cdot\sin\varphi)^2} =$

$x = r\cdot\cos\varphi$
$y = r\cdot\sin\varphi$

$= \lim_{r\downarrow 0} \frac{r^2\cos^2\varphi \cdot \ln(1+\cos\varphi)}{r^2\cdot(\cos^2\varphi+\sin^2\varphi)} =$

$= \lim_{r\downarrow 0} \left(\cos^2\varphi \cdot \ln(1+\cos\varphi)\right) =$

$= \cos^2\varphi \cdot \lim_{r\downarrow 0} \ln(1+r\cdot\underbrace{\cos\varphi}_{\in[-1,1]}) =$ 0

$= \cos^2\varphi \cdot \lim_{r\downarrow 0} \ln 1 =$

$= \underline{\underline{0}}$

ZVEZNOST FUNKCIJ VEČ SPREMENLJIVK

FUNKCIJA f DVEH SPREMENLJIVK JE ZVEZNA V TOČKI (a,b) IZ DEFINICIJSKEGA OBMOČJA, ČE VELJA $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

ANALOGNO VELJA ZA FUNKCIJE VEČ KOT DVEH SPREMENLJIVK.

Določi takšno število d, da bo funkcija $f:\mathbb{R}^2\to\mathbb{R}$ podana z naslednjim predpisom zvezna.

$$f(x,y)=\begin{cases}\frac{xy\sin(x^2+y^2)}{(x^2+y^2)^{\frac{3}{2}}} & : (x,y)\neq(0,0)\\ d & : (x,y)=(0,0)\end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) \underset{\uparrow}{=} f(0,0) = d$$

zveznost v $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{xy\cdot\sin(x^2+y^2)}{(x^2+y^2)^{3/2}} = \lim_{r\downarrow 0}\frac{r\cos\varphi\cdot r\sin\varphi\cdot\sin(r^2)}{(r^2)^{3/\cancel{2}}} =$$

$x=r\cdot\cos\varphi$
$y=r\cdot\sin\varphi$

$$\lim_{r\downarrow 0}\frac{\cos\varphi\cdot\sin\varphi\cdot\sin(r^2)}{r} =$$

$$\lim_{x\to 0}\frac{\sin(x)}{x}$$

$R=r^2$

$r=\sqrt{R}$

$$= \lim_{r\downarrow 0}\frac{r^{\cancel{2}}\cos\varphi\sin\varphi\cdot\sin(r^2)}{\cancel{r}\cdot r^2} =$$

$$= \lim_{R\downarrow 0}\frac{\sqrt{R}\cdot\cos\varphi\cdot\sin\varphi\cdot\sin R}{R} =$$

$\to 1$

$R=r^2$

$r\to 0 \Rightarrow R\to 0$

$$= \lim_{R\to 0}\left(\sqrt{R}\cdot\underbrace{\cos\varphi\cdot\sin\varphi}_{\text{omejeno}}\right) = 0$$

$$d=f(0,0)=0$$

Ali se dajo naslednje funkcije zvezno razširiti v izhodišče?

a) $f(x,y) = \frac{x^3+x^2+y^2}{x^2+y^2}$ b) $f(x,y) = \frac{x^2+2y^2}{2x^2+y^2}$

a) $f(x,y) = \frac{x^3+x^2+y^2}{x^2+y^2}$ $\leftarrow$ v $(0,0)$ ni definirana

$\bar{f}(x,y) = f(x,y)$ za $(x,y) \neq 0$

$\bar{f}(0,0) = \lim\limits_{(x,y)\to(0,0)} f(x,y)$ če obstaja

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\downarrow 0} \frac{r^3\cdot\cos^3\varphi + r^2}{r^2} = \lim_{r\downarrow 0} \frac{\cancel{r^2}(r\cos^3\varphi+1)}{\cancel{r^2}} = \lim_{r\downarrow 0}(\overset{0}{\cancel{r\cdot\cos^3\varphi}}+1) = 1$$

$x = r\cdot\cos\varphi$
$y = r\cdot\sin\varphi$

Razširitev: $\bar{f}(x,y) = \begin{cases} f(x,y) & ; (x,y) \neq (0,0), \\ 1 & ; (x,y) = (0,0). \end{cases}$

b) $f(x,y) = \frac{x^2+2y^2}{2x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\downarrow 0} \frac{\cancel{r^2}\cdot\cos^2\varphi + 2\cdot\cancel{r^2}\cdot\sin^2\varphi}{2\cdot\cancel{r^2}\cdot\cos^2\varphi + \cancel{r^2}\cdot\sin^2\varphi} =$$

$x = r\cdot\cos\varphi$
$y = r\cdot\sin\varphi$

$$= \frac{1+\sin^2\varphi}{1+\cos^2\varphi}$$ (odvisno od kota φ)

$\Rightarrow$ ZVEZNA RAZŠIRITEV NE OBSTAJA

PARCIALNI ODVODI

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$f(x,y) = x^2 + 3xy + \frac{2}{y}$$

$$* \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Parcialni odvodi prvega reda:

$$\frac{\partial f}{\partial x}(x,y) = 2x + 3y + 0$$

$$\frac{\partial f}{\partial y}(x,y) = 3x + 2 \cdot (-1) \cdot y^{-2}$$

Parcialni odvodi drugega reda:

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2 \qquad \frac{\partial^2 f}{\partial y \partial x}(x,y) = 3$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 3 \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = -2 \cdot (-2) \cdot y^{-3} = 4y^{-3}$$

> Naj bo $f(x,y) = \frac{\cos(x-2y)}{\cos(x+2y)}$. Izračunaj $\frac{\partial f}{\partial x}\left(\frac{\pi}{4}, \pi\right)$ in $\frac{\partial f}{\partial y}\left(\frac{\pi}{4}, \pi\right)$.

$$f_x(x,y) = \frac{-\sin(x-2y) \cdot 1 \cdot \cos(x+2y) - (-\sin(x+2y) \cdot 1) \cdot \cos(x-2y)}{(\cos(x+2y))^2}$$

$$f_x\left(\frac{\pi}{4}, \pi\right) = \frac{-\sin(\frac{\pi}{4}-2\pi) \cdot \cos(\frac{\pi}{4}+2\pi) + \sin(\frac{\pi}{4}+2\pi) \cdot \cos(\frac{\pi}{4}-2\pi)}{(\cos(\frac{\pi}{4}+2\pi))^2} = \frac{-\sin\frac{\pi}{4} \cdot \cos\frac{\pi}{4} + \sin\frac{\pi}{4} \cdot \cos\frac{\pi}{4}}{\cos^2\frac{\pi}{4}} = \underline{\underline{0}}$$

$$f_y(x,y) = \frac{-\sin(x-2y) \cdot (-2) \cdot \cos(x+2y) - (-\sin(x+2y) \cdot 2 \cdot \cos(x-2y))}{(\cos(x+2y))^2}$$

$$f_y\left(\frac{\pi}{4}, \pi\right) = \frac{2\sin(\frac{\pi}{4}-2\pi) \cdot \cos(\frac{\pi}{4}+2\pi) + 2 \cdot \sin(\frac{\pi}{4}+2\pi) \cdot \cos(\frac{\pi}{4}-2\pi)}{(\cos(\frac{\pi}{4}+2\pi))^2} = \frac{4 \cdot \sin\frac{\pi}{4} \cdot \cancel{\cos\frac{\pi}{4}}}{(\cos\frac{\pi}{4})^{\cancel{2}}} = 4\,\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \underline{\underline{4}}$$

Izračunaj parcialne odvode prvega in drugega reda funkcij f, podanih z naslednjimi predpisi.

(a) $f(x,y) = \arcsin(2x-y)$

(b) $f(x,y) = y\arctan(xy)$

(c) $f(x,y) = e^{-x}(x-y^2)$

(d) $f(x,y) = y^{xy}$

(a) $f(x,y) = \arcsin(2x-y)$

$$f_x(x,y) = \frac{1}{\sqrt{1-(2x-y)^2}}\cdot 2 = \frac{2}{\sqrt{1-(2x-y)^2}}$$

$$f_y(x,y) = \frac{1}{\sqrt{1-(2x-y)^2}}\cdot(-1) = -\frac{1}{\sqrt{1-(2x-y)^2}}$$

$$f_{xx}(x,y) = \cancel{2}\cdot\left(+\cancel{\tfrac{1}{2}}\right)\cdot\frac{1}{\left(\sqrt{1-(2x-y)^2}\right)^3}\cdot(+2)\cdot(2x-y)\cdot 2 = \frac{4\cdot(2x-y)}{(1-(2x-y)^2)^{3/2}}$$

$$f_{xy}(x,y) = 2\cdot\left(-\tfrac{1}{2}\right)\cdot\frac{1}{(1-(2x-y)^2)^{3/2}}\cdot(-2)\cdot(2x-y)\cdot(-1) = \frac{-2(2x-y)}{(1-(2x-y)^2)^{3/2}}$$

$$f_{yy}(x,y) = +\left(-\cancel{\tfrac{1}{2}}\right)\cdot\frac{1}{(1-(2x-y)^2)^{3/2}}\cdot(-\cancel{2})\cdot(2x-y)\cdot(+1) = \frac{2x-y}{(1-(2x-y)^2)^{3/2}}$$

(b) $f(x,y) = y\arctan(xy)$

$$f_x(x,y) = y\cdot\frac{1}{1+(xy)^2}\cdot y = \frac{y^2}{1+x^2y^2}$$

$$f_y(x,y) = \arctan(xy) + y\cdot\frac{1}{1+(xy)^2}\cdot x$$

$$f_{xx}(x,y) = y^2\cdot\left(-\frac{1}{(1+x^2y^2)^2}\right)\cdot(2x\cdot y^2) = -\frac{2xy^4}{(1+x^2y^2)^2}$$

$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{2y(1+x^2y^2)-y^2\cdot 2x^2y}{(1+x^2y^2)^2} = \frac{2y+\cancel{2x^2y^3}-\cancel{2x^2y^3}}{(1+x^2y^2)^2} = \frac{2y}{(1+x^2y^2)^2}$$

$$f_{yy}(x,y) = \frac{1}{1+(xy)^2}\cdot x + \frac{x}{1+(xy)^2} - y\cdot x\cdot\frac{1}{(1+(xy)^2)^2}\cdot 2x^2y =$$

$$= \frac{2x}{1+x^2y^2} - \frac{2x^3y^2}{(1+x^2y^2)^2} = \frac{1}{(1+x^2y^2)^2}\cdot\left(2x+\cancel{2x^3y^2}-\cancel{2x^3y^2}\right) = \frac{2x}{(1+x^2y^2)^2}$$

$f(x)$	$f'(x)$
c	0
x^2	$2x$
x^m	$m\cdot x^{m-1}$
$\sqrt{x}$	$\frac{1}{2\cdot\sqrt{x}}$
$x^{\frac{1}{m}}$	$\frac{1}{m}\cdot x^{\frac{1}{m}-1}$
x^x	$x^x(1+\ln x)$
e^x	e^x
a^x	$a^x\cdot\ln a$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x\cdot\ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\cot x$	$-\frac{1}{\sin^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$

(c) $f(x,y) = e^{-x}(x-y^2)$

$$f_x(x,y) = -e^{-x}(x-y^2) + e^{-x}\cdot 1 = e^{-x}(-x+y^2+1)$$

$$f_y(x,y) = e^{-x}(-2y)$$

$$f_{xx}(x,y) = -e^{-x}(-x+y^2+1) + e^{-x}(-1) = e^{-x}\left(x-y^2-1-1\right)$$

$$f_{xy}(x,y) = (-2y)\cdot(-e^{-x}) = 2ye^{-x}$$

$$f_{yy}(x,y) = e^{-x}\cdot(-2) = -2\cdot e^{-x}$$

$f(x)$	$f'(x)$
c	0
x^2	$2x$
x^m	$m\cdot x^{m-1}$
$\sqrt{x}$	$\frac{1}{2\cdot\sqrt{x}}$
$x^{\frac{1}{m}}$	$\frac{1}{m}\cdot x^{\frac{1}{m}-1}$
x^x	$x^x(1+\ln x)$
e^x	e^x
a^x	$a^x\cdot\ln a$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x\cdot\ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\cot x$	$-\frac{1}{\sin^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$

(d) $f(x,y) = y^{xy}$ $= \left(\frac{1}{x}\right)^{xy}(xy)^{(xy)} = (y^y)^x$

$$f_x(x,y) = (y^y)^x\cdot\ln(y^y) = y^{xy+1}\cdot\ln y = (y\cdot\ln y)\cdot(y^y)^x$$

$$f_{xx}(x,y) = \underbrace{(y\cdot\ln y)}_{\text{"konstanta"}}\cdot\underbrace{(y\cdot\ln y)\cdot(y^y)^x}_{\frac{\partial}{\partial x}(y^y)^x} = (y\cdot\ln y)^2\cdot y^{xy} = y^{xy+2}\cdot(\ln y)^2$$

$$f_y(x,y) = x\cdot(y^y)^{x-1}\cdot\underbrace{y^y\cdot(1+\ln y)}_{\frac{\partial}{\partial y}(y^y)} = x\cdot(y^y)^{x\cancel{-1+1}}\cdot(1+\ln y) = x\cdot y^{xy}\cdot(1+\ln y)$$

$$f_{yy}(x,y) = x\cdot\underbrace{\left(x\cdot y^{xy}\cdot(1+\ln y)\right)}_{\frac{\partial}{\partial y}(y^{xy})}\cdot(1+\ln y) + x\cdot y^{xy}\cdot\frac{1}{y} =$$

$$= x^2 y^{xy}\cdot(1+\ln y)^2\cdot x\cdot y^{xy-1}$$

$$f_{xy}(x,y) = y^{xy}\cdot(1+\ln y) + x\cdot(1+\ln y)\cdot\underbrace{(y\cdot\ln y)\cdot y^{yx}}_{\frac{\partial}{\partial x}y^{xy}} = y^{xy}(1+\ln y)\cdot(1+xy\ln y)$$

20.10.2020

GRADIENT

$f: \mathbb{R}^2 \to \mathbb{R}$ $\quad \nabla f = (f_x, f_y)$

$f: \mathbb{R}^3 \to \mathbb{R}$ $\quad \nabla f = (f_x, f_y, f_z)$

$\nabla f(x_0, y_0)$ ... smer, v kateri funkcija f v točki (x_0, y_0) najhitreje narašča.

Izračunaj kot med gradientoma funkcije $f(x,y) = \ln \frac{y}{x}$ v točkah $(-1,-1)$ in $(1,1)$.
Opomba: $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$

$f(x,y) = \ln \frac{y}{x} = \ln y - \ln x$

$f_x(x,y) = -\frac{1}{x}$

$f_y(x,y) = \frac{1}{y}$

$\nabla f(x,y) = \left(-\frac{1}{x}, \frac{1}{y}\right)$

$\nabla f(-1,-1) = (1,-1)$

$\nabla f(1,1) = (-1,1)$

Kot med vektorjema:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\varphi$$

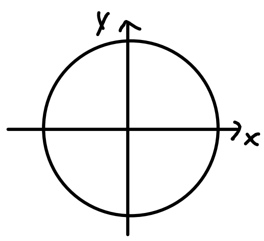
$$(1,-1) \cdot (-1,1) = \|(1,-1)\| \cdot \|(-1,1)\| \cdot \cos\varphi$$

$$-2 = \sqrt{2} \cdot \sqrt{2} \cdot \cos\varphi$$

$$-2 = 2 \cdot \cos\varphi$$

$$\cos\varphi = -1$$

$$\varphi = \pi$$



V katerih točkah (x, y, z) je gradient funkcije

$$f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$$

a) pravokoten na z-os,

b) enak 0.

$f_x(x,y,z) = 3x^2 - 3yz$

$f_y(x,y,z) = 3y^2 - 3xz$

$f_z(x,y,z) = 3z^2 - 3xy$

$\operatorname{grad} f(x,y,z) = \nabla f(x,y,z) = 3 \cdot (x^2 - yz,\ y^2 - xz,\ z^2 - xy)$

a) $\nabla f(x,y,z) \cdot (0,0,1) = 0$

$3 \cdot (x^2 - yz,\ y^2 - xz,\ z^2 - xy) \cdot (0,0,1) = 0$

$z^2 - xy = 0$

$z^2 = xy$

$R = \{(x,y,z);\ z^2 = xy\}$

b) $\nabla f(x,y,z) = \vec{0} = (0,0,0)$

$3 \cdot (x^2 - yz,\ y^2 - xz,\ z^2 - xy) = 0$

$$\begin{aligned} x^2 &= yz \\ y^2 &= xz \\ z^2 &= xy \end{aligned}$$

$x^2 - y^2 = yz - xz$

$(x-y) \cdot (x+y) = z \cdot (y-x)$

$(x-y) \cdot (x+y+z) = 0$

1.) $x = y$: $\left.\begin{aligned} x^2 &= xz \\ x^2 &= xz \\ z^2 &= x^2 \end{aligned}\right\}$ $x = 0, z = 0$ ali $z = x$

$T(0,0,0)$

$T(x,y,z);\ x = y = z$

2.) $x + y + z = 0$

$z = -x - y$

$z^2 = xy$

$z^2 = (x+y)^2 = x^2 + 2xy + y^2$

$xy = x^2 + 2xy + y^2$

$x^2 + xy + y^2 = 0$

$(x + \frac{y}{2})^2 - \frac{y^2}{4} + y^2 = 0$

$(x + \frac{y}{2})^2 + \frac{3}{4}y^2 = 0$

$x + \frac{y}{2} = 0$

$y = 0$

$\Downarrow$

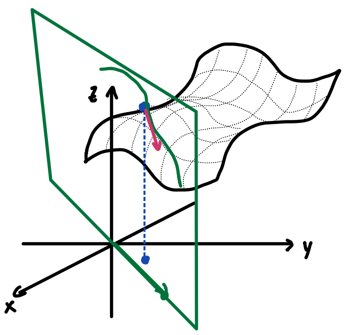
$x = 0$

$\Downarrow$

$z = 0$

$R = \{(x,x,x);\ x \in \mathbb{R}\}$

SMERNI ODVOD



$\vec{n}$... enotski vektor $(\|\vec{n}\| = 1)$

Smerni odvod v smeri vektorja $\vec{n}$:

$$\frac{\partial f}{\partial \vec{n}}(\vec{r}_0) = \lim_{t \to 0} \frac{f(\vec{r}_0 + t\cdot\vec{n}) - f(\vec{r}_0)}{t}$$

$\left(\begin{array}{l} \vec{r}_0 = (x_0, y_0) \text{ pri} \\ f: \mathbb{R}^2 \to \mathbb{R} \end{array}\right)$

$$\frac{\partial f}{\partial \vec{n}}(\vec{r}) = \vec{n} \cdot \operatorname{grad}(\vec{r})$$

Posebna primera: $\vec{n} = (1,0): \frac{\partial f}{\partial \vec{n}} = f_x$

$\vec{n} = (0,1): \frac{\partial f}{\partial \vec{n}} = f_y$

Izračunajte smerni odvod funkcije $f(x,y) = \frac{x^2}{4} + \frac{y^2}{9}$ v točki $(2,0)$ v smeri krajevnega vektorja te točke.

$$\vec{n} = \frac{(2,0)}{\|(2,0)\|} = \frac{(2,0)}{2} = (1,0)$$

$$f_x = \frac{2x}{4} = \frac{x}{2}$$

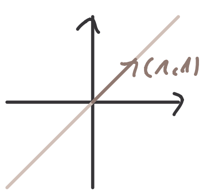
$$f_y = \frac{2y}{9}$$

$$\nabla f = \left(\frac{x}{2}, \frac{2y}{9}\right)$$

$$\frac{\partial f}{\partial \vec{n}}(x,y) = \vec{n} \cdot \nabla f(x,y) = (1,0)\cdot\left(\frac{x}{2}, \frac{2y}{9}\right) = \frac{x}{2}$$

$$\frac{\partial f}{\partial \vec{n}}(2,0) = \frac{2}{2} = 1$$

Izračunajte smerni odvod funkcije $f(x,y) = \arctan(xy)$ v točki $(1,1)$ v smeri simetrale prvega kvadranta.



$$\vec{n} = \frac{(1,1)}{\|(1,1)\|} = \frac{(1,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\nabla f(x,y) = \frac{1}{1+x^2y^2} \cdot \begin{bmatrix} y \\ x \end{bmatrix}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{1+x^2y^2} \cdot y$$

$$\nabla f(1,1) = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{1+x^2y^2} \cdot x$$

$$\frac{\partial f}{\partial \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{2}, \frac{1}{2}\right) = \underline{\underline{\frac{1}{\sqrt{2}}}}$$

LINEARNI PRIBLIŽKI IN DIFERENCIABILNOST

$f: \mathbb{R}^2 \to \mathbb{R}$ je v točki (a,b) DIFERENCIABILNA, če je parcialno odvedljiva in če velja:

$$f(a+h, b+k) = f(a,b) + f_x(a,b)\cdot h + f_y(a,b)\cdot k + R(h,k),$$

kjer gre $\frac{R(h,k)}{\sqrt{h^2+k^2}}$ proti 0, ko gre (h,k) proti $(0,0)$.

Linearni približek:

$$f(a+h, b+k) \approx f(a,b) + f_x(a,b)\cdot h + f_y(a,b)\cdot k$$

S pomočjo diferenciala izračunajte približke vrednosti

a) $\sqrt{4.05^2 + 2.93^2}$

b) $e^{0.02} \cdot 0.97$

c) $\frac{\sqrt[3]{8.06}}{24.5}$

d) $\ln(\sqrt[3]{1.03} + \sqrt[4]{0.98} - 1)$

a) $\sqrt{4.05^2 + 2.93^2}$

$$f(x,y) = \sqrt{x^2+y^2}$$

$$f_x(x,y) = \cancel{\tfrac{1}{2}}\frac{1}{\sqrt{x^2+y^2}}\cdot \cancel{2}x$$

$$f_y(x,y) = \cancel{\tfrac{1}{2}}\frac{1}{\sqrt{x^2+y^2}}\cdot \cancel{2}y$$

Znamo izračunat:

$$f(4,3) = \sqrt{4^2+3^2} = \sqrt{25} = 5$$

$$f_x(4,3) = \frac{1}{5}\cdot 4 = \frac{4}{5}$$

$$f_y(4,3) = \frac{1}{5}\cdot 3 = \frac{3}{5}$$

$$f(4.05, 2.93) = f(\overset{a\,+\,h}{4+0.05}, \overset{b\,+\,k}{3-0.07}) \approx$$

$$\approx f(4,3) + f_x(4,3)\cdot 0.05 + f_y(4,3)\cdot(-0.07) =$$

$$= 5 + \frac{4}{5}\cdot 0.05 + \frac{3}{5}\cdot(-0.07) =$$

$$= 5 + 0.04 - 3\cdot 0.014 = 5 + 0.04 - 0.042 = \underline{\underline{4.998}}$$

d) $\ln(\sqrt[3]{1.03} + \sqrt[4]{0.98} - 1)$

Znamo izračunat:

$$f(x,y) = \ln(\sqrt[3]{x} + \sqrt[4]{y} - 1) \qquad f(1,1) = \ln(1+\cancel{1}-\cancel{1}) = \ln 1 = 0$$

$$f_x(x,y) = \frac{1}{\sqrt[3]{x}+\sqrt[4]{y}-1} \cdot \frac{1}{3} \cdot x^{-\frac{2}{3}} \qquad f_x(1,1) = \frac{1}{1+\cancel{1}-\cancel{1}} \cdot \frac{1}{3} \cdot 1^{-\frac{2}{3}} = \frac{1}{3}$$

$$f_y(x,y) = \frac{1}{\sqrt[3]{x}+\sqrt[4]{y}-1} \cdot \frac{1}{4} \cdot y^{-\frac{3}{4}} \qquad f_y(1,1) = \frac{1}{1+\cancel{1}-\cancel{1}} \cdot \frac{1}{4} \cdot 1^{-\frac{3}{4}} = \frac{1}{4}$$

$$f(1.03, 0.98) = f(1+0.03, 1-0.02) \approx$$

$$\approx f(1,1) + f_x(1,1)\cdot 0.03 + f_y(1,1)\cdot(-0.02) =$$

$$= 0 + \frac{1}{3}\cdot 0.03 - \frac{1}{4}\cdot 0.02 =$$

$$= 0.01 - 0.005 =$$

$$= 0.005$$

Izračunajte totalni diferencial funkcije $f(x,y) = \frac{x\cdot y}{x-y}$.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x-y)-xy}{(x-y)^2} = \frac{\cancel{xy}-y^2-\cancel{xy}}{(x-y)^2} = -\frac{y^2}{(x-y)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x(x-y)-xy\cdot(-1)}{(x-y)^2} = \frac{x^2-\cancel{xy}+\cancel{xy}}{(x-y)} = \frac{x^2}{(x-y)^2}$$

$$df = -\frac{y^2}{(x-y)^2}dx + \frac{x^2}{(x-y)^2}dy$$

Za naslednje funkcije izračunajte linearno aproksimacijo v dani točki.

a) $f(x,y) = x^2 + y$ v $(x_0, y_0) = (\frac{1}{4}, \frac{3}{4})$

b) $f(r,\varphi) = r\cos(\varphi)$ v $(r_0, \varphi_0) = (1, \pi)$

a) $f(x,y) = x^2 + y$ v $(x_0, y_0) = (\frac{1}{4}, \frac{3}{4})$

$$f_x(x,y) = 2x \qquad x = \tfrac{1}{4} + h$$

$$f_y(x,y) = 1 \qquad y = \tfrac{3}{4} + k$$

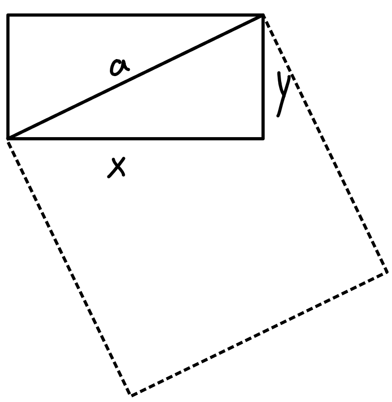
$$f(x,y) \approx f(\tfrac{1}{4}, \tfrac{3}{4}) + f_x(\tfrac{1}{4}, \tfrac{3}{4}) \cdot (x - \tfrac{1}{4}) + f_y(\tfrac{1}{4}, \tfrac{3}{4}) \cdot (y - \tfrac{3}{4}) =$$

$$= (\tfrac{1}{4})^2 + \tfrac{3}{4} + \tfrac{2}{4} \cdot (x - \tfrac{1}{4}) + 1 \cdot (y - \tfrac{3}{4}) =$$

$$= \tfrac{1}{16} + \cancel{\tfrac{3}{4}} + \tfrac{1}{2}x - \tfrac{2}{16} + y - \cancel{\tfrac{3}{4}} =$$

$$= \tfrac{1}{2}x + y - \tfrac{1}{16}$$

Približno izračunajte ploščino kvadrata, ki ima za stranico diagonalo pravokotnika s stranicama 3.03 cm in 3.97 cm.



$$f(x,y) = x^2 + y^2 \qquad f(3,4) = 3^2 + 4^2 = 25$$

$$f_x(x,y) = 2x \qquad f_x(3,4) = 2 \cdot 3 = 6$$

$$f_y(x,y) = 2y \qquad f_y(3,4) = 2 \cdot 4 = 8$$

$$f(3.03, 3.97) = f(3 + 0.03, 4 - 0.03) \approx$$

$$\approx f(3,4) + f_x(3,4) \cdot 0.03 + f_y(3,4) \cdot (-0.03) =$$

$$= 25 + 6 \cdot 0.03 + 8 \cdot (-0.03) =$$

$$= 25 + 0.18 - 0.24 =$$

$$= 24.94$$

$$a = \sqrt{x^2 + y^2}$$

$$p = a^2 = x^2 + y^2$$

Naj bo

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}; & (x,y) \neq (0,0) \\ a; & (x,y) = (0,0). \end{cases}$$

a) Določite a tako, da bo f zvezna v $(0,0)$.

b) Izračunajte parcialne odvode, kjer obstajajo.

c) Ali je f v izhodišču parcialno odvedljiva?

d) Ali je f v izhodišču diferenciabilna?

e) Ali so v izhodišču parcialni odvodi f zvezni?

f) Ali je f v izhodišču zvezno parcialno odvedljiva?

a.) $a = \lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim\limits_{r\to 0} \frac{r^2 \cdot \cos\varphi \cdot \sin\varphi}{r} = 0$

b.) $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{y\sqrt{x^2+y^2} - xy\,\cancel{\tfrac{1}{2}}\,\frac{1}{\sqrt{x^2+y^2}}\cdot \cancel{2}x}{x^2+y^2} = \frac{x^2y + y^3 - x^2y}{(x^2+y^2)^{3/2}} = \frac{y^3}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{x^3}{(x^2+y^2)^{3/2}}$$

$(x,y) = (0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{0}{|h|} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{0}{|h|} - 0}{h} = 0$$

Pazi! računanje odvodov po definiciji

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{y^3}{(x^2+y^2)^{3/2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases} \qquad \frac{\partial f}{\partial y} = \begin{cases} \frac{x^3}{(x^2+y^2)^{3/2}} & ; (x,y) \neq 0 \\ 0 & ; (x,y) = 0 \end{cases}$$

c.) JA

d) $f(a+h,b+k)=f(a,b)+f_x(a,b)\cdot h+f_y(a,b)\cdot k+R(h,k)$ 30.10.2020

$(a,b)=(0,0)$:

$$f(h,k)=f(0,0)+f_x(0,0)\cdot h+f_y(0,0)\cdot k+R(h,k)$$

$(h,k)\neq(0,0)$:

$$R(h,k)=f(h,k)-f(0,0)-f_x(0,0)\cdot h-f_y(0,0)\cdot k=$$

$$=\frac{h\cdot k}{\sqrt{h^2+k^2}}-0-0\cdot h-0\cdot k=$$

$$=\frac{h\cdot k}{\sqrt{h^2+k^2}}$$

PAZI! na vajah naredili napako ($f_x(0,0)=0$ in $f_y(0,0)=0$)

$$\lim_{(h,k)\to(0,0)}\frac{R(h,k)}{\sqrt{h^2+k^2}}=\lim_{(h,k)\to(0,0)}\frac{\frac{hk}{\sqrt{h^2+k^2}}}{\sqrt{h^2+k^2}}=$$

$$=\lim_{(h,k)\to(0,0)}\frac{hk}{h^2+k^2}=$$

$$=\lim_{r\to 0}\frac{r\cdot\cos\varphi\cdot r\cdot\sin\varphi}{r^2\cdot\cos^2\varphi+r^2\cdot\sin^2\varphi}=$$

$$=\lim_{r\to 0}\frac{\cancel{r^2}\cdot(\cos\varphi\cdot\sin\varphi)}{\cancel{r^2}}=\cos\varphi\cdot\sin\varphi$$

$\Rightarrow \lim\limits_{(h,k)\to(0,0)}\frac{R(h,k)}{\sqrt{h^2+k^2}}\neq 0$ $\Rightarrow$ f v izhodišču ni diferenciabilna

e.) f diferenciabil $\Rightarrow$ f parcialno odvedljiva

f parcialno zvezno odvedljiva $\Rightarrow$ f diferenciabilna

Če bi f bila parcialno zvezno odvedljiva, bi bila tudi diferenciabilna.

$\Rightarrow$ f ni parcialno zvezno odvedljiva

*NA PREDAVANJIH

Naj bo

$$f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0).\end{cases}$$

a) Ali je f zvezna?

b) Ali je f parcialno odvedljiva?

c) Ali je f v izhodišču diferenciabilna?

d) Izračunajte parcialne odvode drugega reda funkcije f v izhodišču.

e) Ali je f v izhodišču zvezno parcialno odvedljiva?

a) $$\lim_{(h,k)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = \lim_{r\downarrow 0} \frac{r\cos\varphi\cdot r\sin\varphi\,(r^2\cos^2\varphi-r^2\sin^2\varphi)}{r^2\cos^2\varphi+r^2\sin^2\varphi} =$$

$$= \lim_{r\downarrow 0} \frac{\cancel{r^2}\cdot\frac{1}{2}\cdot\sin 2\varphi\cdot r^2\cdot\cos 2\varphi}{\cancel{r^2}} =$$

$$= \lim_{r\downarrow 0} \frac{1}{2}r^2\cdot\underbrace{\sin 2\varphi\cdot\cos 2\varphi}_{\text{omejeno}} = 0$$

*b) $(x,y)\neq(0,0)$:

$$\frac{\partial f}{\partial x}(x,y) = \frac{\left(y(x^2-y^2)+xy\cdot 2x\right)\cdot(x^2+y^2)-xy(x^2-y^2)\cdot 2x}{(x^2+y^2)^2} =$$

$$= \frac{(yx^2-y^3+2yx^2)\cdot(x^2+y^2)-2yx^2(x^2-y^2)}{(x^2+y^2)^2} = \frac{y(3x^2-y^2)(x^2+y^2)-2yx^2(x^2-y^2)}{(x^2+y^2)^2} =$$

$$= \frac{y\cdot(3x^4+3x^2y^2-x^2y^2-y^4-2x^4+2x^2y^2)}{(x^2+y^2)^2} = \frac{y\cdot(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) \underset{\text{podobno}}{=} -\frac{x(y^4+4x^2y^2-x^4)}{(x^2+y^2)^2}$$

$(x,y)=(0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = 0$$

Odgovor: JA

c) $R(h,k) = f(0+h,0+k) - f(0,0) - f_x(0,0)\cdot h - f_y(0,0)\cdot k$

$$= f(h,k) - 0 - 0\cdot h - 0\cdot k = f(h,k) = hk\,\frac{h^2-k^2}{h^2+k^2}$$

$$\lim_{(h,k)\to(0,0)} \frac{R(h,k)}{\sqrt{h^2+k^2}} = \lim_{r\to 0} \frac{1}{\sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi}} \cdot r\cos\varphi \cdot r\sin\varphi \cdot \frac{\cancel{r^2}(\cos^2\varphi - \sin^2\varphi)}{\cancel{r^2}(\cos^2\varphi+\sin^2\varphi)} =$$

$$= \lim_{r\to 0} \frac{1}{\sqrt{r^2}} \cdot r^2 \cdot \frac{1}{2}\sin 2\varphi \cdot \cos 2\varphi = \lim_{r\to 0} \frac{r^2}{\sqrt{r^2}} \cdot \frac{1}{4} \cdot \sin 4\varphi = 0$$

* d) $\frac{\partial^2 f}{\partial x^2}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$

$$\frac{\partial^2 f}{\partial y^2}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(0,h) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial y\partial x}(0,0) &= \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{-\frac{h^5}{h^4} - 0}{h} = -1 \\ \frac{\partial^2 f}{\partial x\partial y}(0,0) &= \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0} -\frac{\frac{h^5}{h^4} - 0}{h} = 1 \end{aligned}\right\} \frac{\partial^2 f}{\partial y\partial x}(x,y) \neq \frac{\partial^2 f}{\partial x\partial y}(x,y)$$

e) Trditev: Če parcialna odvoda drugega reda obstajata in sta <u>zvezni</u> funkciji potem sta enaka

$\frac{\partial^2 f}{\partial x\partial y} \neq \frac{\partial^2 f}{\partial y\partial x} \Rightarrow$ mešana odvoda nista zvezna

$\Rightarrow f$ ni parcialno odvedljiva

POSREDNO ODVAJANJE

VERIŽNO PRAVILO

$$f(x,y) = f(z(x,y)) \quad \Rightarrow \quad \frac{\partial f}{\partial x} = \frac{df}{dz} \cdot \frac{\partial z}{\partial x}, \qquad \frac{\partial f}{\partial y} = \frac{df}{dz} \cdot \frac{\partial z}{\partial y}$$

TOTALNI ODVOD

$$w(t) = w(x(t), y(t)) \quad \Rightarrow \quad \frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

ZAMENJAVA KOORDINAT

$$g(x,y) = f(u(x,y), v(x,y)) \quad \Rightarrow \quad \frac{\partial g}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x}, \qquad \frac{\partial g}{\partial y} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial y}$$

Funkcije več spremenljivk:

$$[f(g_1(t), \dots, g_n(t))]' = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(g_1(t), \dots, g_n(t)) \cdot g_j'(t)$$

S pomočjo posrednega odvajanja izračunajte odvod funkcije w, podane s predpisom $w(t) = t^{\cos(t)}$.

$$w(t) = t^{\cos t} = w(x(t), y(t))$$

$$w(x,y) = x^y \quad ; \quad x(t) = t, \quad y(t) = \cos t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} = (y \cdot x^{y-1}) \cdot 1 + x^y \cdot \ln x \cdot (-\sin t) =$$

$$= \cos t \cdot t^{\cos t - 1} + t^{\cos t} \cdot \ln t \cdot (-\sin t)$$

$$w'(t) = t^{\cos t} \cdot \left(\frac{\cos t}{t} - \sin t \cdot \ln t \right)$$

Naj bo $w = x^2 e^{\frac{y}{x}}$ ter naj bo nadalje $x(t) = \frac{1-t^2}{1+t^2}$ in $y(t) = \frac{2t}{1+t^2}$. Izračunajte $\frac{dw}{dt}$.

DN?

$$w(t) = w(x(t), y(t))$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial w}{\partial x} = 2xe^{\frac{y}{x}} + \cancel{x^2} \cdot e^{\frac{y}{x}} \cdot \left(-\frac{y}{\cancel{x^2}}\right) = e^{\frac{y}{x}} (2x-y)$$

$$\frac{\partial w}{\partial y} = x^2 \cdot e^{\frac{y}{x}} \cdot \frac{1}{x} = x \cdot e^{\frac{y}{x}}$$

$$\frac{dx}{dt} = \frac{-2t(1+t^2)-(1-t^2)\cdot 2t}{(1+t^2)^2} = \frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2\cdot(1+t^2)-2t\cdot 2t}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dw}{dt} = e^{\frac{y}{x}}(2x-y)\cdot\left(-\frac{4t}{(1+t^2)^2}\right) + x\cdot e^{\frac{y}{x}} \cdot \frac{2\cdot(1-t^2)}{(1+t^2)^2} =$$

$$= e^{\frac{y}{x}} \cdot \frac{2}{(1+t^2)^2} \cdot \left(2t\cdot(y-2x) + x\cdot(1-t^2)\right) =$$

$$= e^{\frac{2t}{1-t^2}} \cdot \frac{2}{(1+t^2)^2} \cdot \left(2t\cdot\left(\frac{2t}{1+t^2} - 2\cdot\frac{1-t^2}{1+t^2}\right) + \frac{1-t^2}{1+t^2}\cdot(1-t^2)\right) =$$

$$= e^{\frac{2t}{1-t^2}} \cdot \frac{2}{(1+t^2)^3} \left(4t^2 - 4t(1-t^2) + (1-t^2)^2\right) =$$

$$= e^{\frac{2t}{1-t^2}} \cdot \frac{2}{(1+t^2)^3} \left(4t^2 - 4t + 4t^3 + 1 - 2t^2 + t^4\right) =$$

$$= e^{\frac{2t}{1-t^2}} \cdot \frac{2}{(1+t^2)^3} \left(t^4 + 4t^3 + 2t^2 - 4t + 1\right) =$$

$$= e^{\frac{2t}{1-t^2}} \cdot \frac{2}{(1+t^2)^3} \cdot (1-2t-t^2)^2$$

Naj bo $w = \frac{y}{x}$ ter naj bo nadalje $x(t) = e^t$ in $y(t) = 1 - e^{2t}$. Izračunajte $\frac{dw}{dt}$.

$$w(t) = w(x(t), y(t))$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial w}{\partial x} = -\frac{y}{x^2} \qquad \frac{dx}{dt} = e^t$$

$$\frac{\partial w}{\partial y} = \frac{1}{x} \qquad \frac{dy}{dt} = -2e^{2t}$$

$$\frac{dw}{dt} = -\frac{y}{x^2} \cdot e^t + \frac{1}{x} \cdot \left(-2e^{2t}\right) = -\frac{1-e^{2t}}{(e^t)^2} \cdot \cancel{e^t} - \frac{2}{\cancel{e^t}} \cdot e^{\cancel{2}t} =$$

$$= -\frac{1}{e^t} + e^t - 2e^t = -\frac{1}{e^t} - e^t$$

Naj bo spremenljivka w odvedljiva funkcija spremenljivke $z = x^2 + y^2$. Izračunajte $y\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y}$

$$w(x,y) = w(z(x,y))$$

$$z(x,y) = x^2 + y^2$$

$$\frac{\partial w}{\partial x} = \frac{dw}{dz} \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{dw}{dz} \cdot \frac{\partial z}{\partial y}$$

$$y \cdot \frac{\partial w}{\partial x} - x \cdot \frac{\partial w}{\partial y} = y \cdot \frac{dw}{dz} \cdot 2x - x \cdot \frac{dw}{dz} \cdot 2y = 0$$

Spremenljivka w naj bo diferenciabilna funkcija spremenljivk x in y.

a) Izrazite parcialna odvoda po polarnih koordinatah s kartezijskimi koordinatami in parcialnima odvodoma, t. j. $\frac{\partial w}{\partial r}$ in $\frac{\partial w}{\partial \varphi}$ izrazite z x, y, $\frac{\partial w}{\partial x}$ in $\frac{\partial w}{\partial y}$.

b) Izrazite $\frac{\partial w}{\partial x}$ in $\frac{\partial w}{\partial y}$ z r, φ, $\frac{\partial w}{\partial r}$ in $\frac{\partial w}{\partial \varphi}$.

a) $w = w(x,y)$

ZAMENJAVA KOORDINAT

$$w(r,\varphi) = w(x(r,\varphi), y(r,\varphi))$$

$$\left.\begin{aligned} x(r,\varphi) &= r\cdot\cos\varphi \\ y(r,\varphi) &= r\cdot\sin\varphi \end{aligned}\right\} \quad r = \sqrt{x^2+y^2}$$

$$\cos\varphi = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\varphi = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial w}{\partial x}\cdot\cos\varphi + \frac{\partial w}{\partial y}\cdot\sin\varphi = \frac{\partial w}{\partial x}\cdot\frac{x}{r} + \frac{\partial w}{\partial y}\cdot\frac{y}{r} = \frac{\partial w}{\partial x}\cdot\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial w}{\partial y}\cdot\frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial w}{\partial \varphi} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \varphi} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \varphi} = \frac{\partial w}{\partial x}\cdot(-r\cdot\sin\varphi) + \frac{\partial w}{\partial y}\cdot r\cdot\cos\varphi = -\frac{\partial w}{\partial x}\cdot y + \frac{\partial w}{\partial y}\cdot x$$

b) $w = w(r,\varphi)$

DN?

ZAMENJAVA KOORDINAT

$$r = \sqrt{x^2+y^2}$$

$$\varphi = \pm\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) \qquad +:\ \varphi\in[0,\pi] \qquad -:\ \varphi\in[-\pi,0]$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$w(r,\varphi) = w(r(x,y), \varphi(x,y))$$

$$\frac{\partial r}{\partial x} = \frac{\cancel{1}}{\cancel{2}}\frac{1}{\sqrt{x^2+y^2}}\cdot \cancel{2}x = \frac{\cancel{1}}{\cancel{r}}\cdot \cancel{r}\cos\varphi = \cos\varphi$$

$$\frac{\partial r}{\partial y} = \frac{\cancel{1}}{\cancel{2}}\frac{1}{\sqrt{x^2+y^2}}\cdot \cancel{2}y = \frac{\cancel{1}}{\cancel{r}}\cdot \cancel{r}\sin\varphi = \sin\varphi$$

$$\frac{\partial \varphi}{\partial x} = \overset{\varphi\in[0,\pi]}{\underset{\varphi\in[-\pi,0]}{\pm}}\left(-\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\right)\cdot \frac{\sqrt{x^2+y^2}-x\cdot\frac{\cancel{1}}{\cancel{2}}\cdot\frac{1}{\sqrt{x^2+y^2}}\cdot\cancel{2}x}{x^2+y^2} =$$

$$= \mp\frac{1}{\sqrt{\frac{x^2+y^2-x^2}{x^2+y^2}}}\cdot\frac{\cancel{x^2}+y^2-\cancel{x^2}}{(x^2+y^2)^{3/2}} =$$

$$= \mp\frac{\cancel{\sqrt{x^2+y^2}}}{\sqrt{y^2}}\cdot\frac{y^2}{\cancel{\sqrt{x^2+y^2}}\cdot(x^2+y^2)} = \mp\frac{y^2}{|y|}\cdot\frac{1}{(x^2+y^2)} =$$

$\varphi\in[0,\pi]: |y| = y$
$\varphi\in[-\pi,0]: |y| = -y$

$$= \mp\frac{y^2}{\pm y}\cdot\frac{1}{r^2} = -\frac{\cancel{y^2}}{\cancel{y}}\cdot\frac{1}{r^2} = -\frac{y}{r^2} = -\frac{\cancel{r}\sin\varphi}{r^{\cancel{2}}} = -\frac{\sin\varphi}{r}$$

$$\frac{\partial \varphi}{\partial y} = \overset{\varphi\in[0,\pi]}{\underset{\varphi\in[-\pi,0]}{\pm}}\left(-\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\right)\cdot\left(-\frac{\cancel{1}}{\cancel{2}}\right)\cdot\frac{x}{\left(\sqrt{x^2+y^2}\right)^3}\cdot\cancel{2}y =$$

$$= \pm\frac{\sqrt{x^2+y^2}}{\sqrt{y^2}}\cdot\frac{xy}{\left(\sqrt{x^2+y^2}\right)^3} = \pm\frac{xy}{|y|}\cdot\frac{1}{x^2+y^2} =$$

$\varphi\in[0,\pi]: |y| = y$
$\varphi\in[-\pi,0]: |y| = -y$

$$= \pm\frac{xy}{\pm y}\cdot\frac{1}{r^2} = \frac{x\cancel{y}}{\cancel{y}}\cdot\frac{1}{r^2} = \frac{x}{r^2} = \frac{\cancel{r}\cos\varphi}{r^{\cancel{2}}} = \frac{\cos\varphi}{r}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial r} = \frac{\partial w}{\partial r}\cdot\cos\varphi + \frac{\partial w}{\partial \varphi}\cdot\left(-\frac{\sin\varphi}{r}\right)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y} = \frac{\partial w}{\partial r}\cdot\sin\varphi + \frac{\partial w}{\partial \varphi}\cdot\left(\frac{\cos\varphi}{r}\right)$$

TAYLORJEVA VRSTA

Funkcije ene spremenljivke:

$$f(x) = f(a) + \frac{f'(a)}{1!}\cdot(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots$$

Funkcije dveh spremenljivk:

$$f(x,y) = f(a,b) + \frac{1}{1!}\cdot\left(f_x(a,b)\cdot(x-a)^1 + f_y(a,b)\cdot(y-b)^1\right) +$$

$$+\frac{1}{2!}\left(f_{xx}(a,b)\cdot(x-a)^2 + 2\cdot f_{xy}(a,b)\cdot(x-a)^1\cdot(y-b)^1 + f_{yy}(a,b)\cdot(y-b)^2\right) +$$

$$+\frac{1}{3!}\left(f_{xxx}(a,b)\cdot(x-a)^3 + 3\cdot f_{xxy}(a,b)\cdot(x-a)^2\cdot(y-b) + 3\cdot f_{xyy}(a,b)\cdot(x-a)\cdot(y-b)^2 + f_{yyy}(a,b)\cdot(y-b)^3\right) +$$

$$+\dots$$

Znane:

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ za vse x

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ za vse x

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ za vse x

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ za $|x|<1$

$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ za $|x|<1$ in $\alpha \in \mathbb{R}$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ za $|x|<1$

Funkcijo $f(x,y) = 2x^2 - xy - y^2 - 6x - 3y + 5$ razvijte v Taylorjevo vrsto okrog točke $(1,-2)$.

$f_x(x,y) = 4x - y - 6$

$f_y(x,y) = -x - 2y - 3$

$f_{xx}(x,y) = 4$

$f_{xy}(x,y) = -1$

$f_{yy}(x,y) = -2$

Vsi parcialni odvodi višjega reda so enaki 0.

$f(1,-2) = \cancel{2} + \cancel{2} - \cancel{4} - \cancel{6} + \cancel{6} + 5 = 5$

$f_x(1,-2) = 4 + 2 - 6 = 0$

$f_y(1,-2) = -1 + 4 - 3 = 0$

$$f(x,y) = f(1,-2) + \frac{1}{1!}\left(f_x(1,-2)\cdot(x-1) + f_y(1,-2)\cdot(y+2)\right)$$
$$+ \frac{1}{2!}\left(f_{xx}(1,-2)\cdot(x-1)^2 + 2\cdot f_{xy}(1,-2)\cdot(x-1)\cdot(y+2) + f_{yy}(1,-2)\cdot(y+2)^2\right) +$$
$$+ \frac{1}{3!}\left(f_{xxx}(1,-2)\cdot(x-1)^3 + \dots\right) + \dots =$$

$$= 5 + 0\cdot(y+2) + \frac{1}{2}\cdot\left(4\cdot(x-1)^2 + 2\cdot(-1)\cdot(x-1)\cdot(y+2) - 2\cdot(y+2)^2\right) =$$

$$= 5 + 2\cdot(x-1)^2 - (x-1)(y+2) - (y+2)^2$$

Dana je funkcija $f(x,y) = \frac{\sqrt[3]{x}}{\sqrt{y}}$. S pomočjo Taylorjevega polinoma drugega reda približno izračunajte $\frac{\sqrt[3]{8.06}}{\sqrt{24.5}}$.

$f(x,y) = x^{\frac{1}{3}} y^{-\frac{1}{2}}$ $\qquad$ $f(8,25) = \frac{2}{5}$

$f_x(x,y) = \frac{1}{3} \cdot x^{-\frac{2}{3}} \cdot y^{-\frac{1}{2}}$ $\qquad$ $f_x(8,25) = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} = \frac{1}{60}$

$f_y(x,y) = -\frac{1}{2} \cdot x^{\frac{1}{3}} y^{-\frac{3}{2}}$ $\qquad$ $f_y(8,25) = -\cancel{\frac{1}{2}} \cdot \cancel{2} \cdot \frac{1}{125} = -\frac{1}{125}$

$f_{xx}(x,y) = \frac{1}{3}\left(-\frac{2}{3}\right) x^{-\frac{5}{3}} y^{-\frac{1}{2}}$ $\qquad$ $f_{xx}(8,25) = \frac{1}{3} \cdot \left(-\frac{2}{3}\right) \cdot \frac{1}{2^5} \cdot \frac{1}{5} = -\frac{1}{9 \cdot 16 \cdot 5}$

$f_{xy}(x,y) = -\frac{1}{2} \cdot \frac{1}{3} \cdot x^{-\frac{2}{3}} \cdot y^{-\frac{3}{2}}$ $\qquad$ $f_{xy}(8,25) = -\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5^3}$

$f_{yy}(x,y) = -\frac{1}{2} \cdot \left(-\frac{3}{2}\right) \cdot x^{\frac{1}{3}} \cdot y^{-\frac{5}{2}}$ $\qquad$ $f_{yy}(8,25) = \cancel{\frac{1}{2}} \cdot \frac{3}{2} \cdot \cancel{2} \cdot \frac{1}{5^5}$

$$f(8.06, 24.5) \approx f(8,25) + \frac{1}{60} \cdot (0.06) - \frac{1}{5^3} \cdot (-0.5) +$$

$$+ \frac{1}{2} \cdot \left(-\frac{1}{9 \cdot 16 \cdot 5} \cdot (0.06)^2 + 2 \cdot \left(-\frac{1}{2^3} \cdot \frac{1}{5^3} \cdot \frac{1}{3}\right) \cdot (0.06) \cdot (-0.5) + \frac{3}{2} \cdot \frac{1}{5^5} \cdot (-0.5)^2 \right) =$$

$$= \frac{2}{5} + \frac{1}{1000} + \frac{1}{250} + \frac{1}{2} \cdot \Bigg(\underbrace{-\frac{1}{1000} \cdot \frac{1}{4 \cdot 5}}_{-0.000005} + \underbrace{\frac{1}{2} \cdot \frac{1}{5^2} \cdot \frac{1}{1000}}_{+0{,}00002} + \underbrace{\frac{3}{2} \cdot \frac{1}{5^3} \cdot \frac{1}{100}}_{+0{,}00012} \Bigg) =$$

$$= 0{,}405 + \frac{1}{2} \cdot 0.000135 = 0{,}405 + 0{,}0000675 = \underline{\underline{0{,}4050675}}$$

2. NAČIN:

3.11.2020

$$T_2(x,y) = c_{00} + c_{10}(x-8) + c_{01}(y-25) + c_{20}(x-8)^2 + c_{11}(x-8)(y-25) + c_{02}(y-25)^2 \; ; \quad c_{ij} \text{... koeficienti v Taylorjevi vrsti}$$

Binomska vrsta: $(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ za $|x| < 1$

$$\binom{\alpha}{n} = \frac{\alpha \cdot (\alpha-1) \cdot \ldots \cdot (\alpha-n+1)}{n!}$$

$$f(x,y) = (8+x-8)^{\frac{1}{3}}(25+y-25)^{-\frac{1}{2}} =$$

$$= 2 \cdot \left(1 + \frac{x-8}{8}\right)^{\frac{1}{3}} \cdot \frac{1}{5}\left(1 + \frac{y-25}{25}\right)^{-\frac{1}{2}} =$$

$$= \frac{2}{5}\left(\sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} \frac{(x-8)^n}{8^n}\right)\left(\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(y-25)^n}{25^n}\right) =$$

$$= \frac{2}{5}\left(1 + \frac{1}{24}(x-8) - \frac{1}{9 \cdot 64}(x-8)^2 + \ldots\right)\left(1 - \frac{1}{50}(y-25) + \frac{3}{8 \cdot 25^2}(y-25)^2 + \ldots\right) =$$

$$\binom{\frac{1}{3}}{2} = \frac{\frac{1}{3} \cdot \left(-\frac{2}{3}\right)}{2} = -\frac{1}{9} \qquad \binom{-\frac{1}{2}}{2} = \frac{-\frac{1}{2} \cdot \left(-\frac{3}{2}\right)}{2} = \frac{3}{8}$$

$$T_2(x,y) = \frac{2}{5}\left(1 - \frac{1}{50}(y-25) + \frac{3}{8 \cdot 25^2}(y-25)^2 + \frac{1}{24}(x-8) - \frac{1}{24 \cdot 50}(x-8)(y-25) - \frac{1}{9 \cdot 64}(x-8)^2\right)$$

$$T_2(8.06, 29.5) = 0.4050675$$

Naj bo $f(x,y) = x \cdot \cos(xy^2)$.

a) Razvijte f v Taylorjevo vrsto okrog točke $(0,0)$.

b) Izračunaj $\frac{\partial^7 f}{\partial x^3 \partial y^4}(0,0)$, $\frac{\partial^{61} f}{\partial x^{21} \partial y^{40}}(0,0)$ in $\frac{\partial^{62} f}{\partial x^{22} \partial y^{40}}(0,0)$

* podobno kot na predavanjih

a) $\cos t = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k x^{2k}$

uporabimo $\cos t$ za $t = xy^2$:

razvoj $\cos(xy^2)$ okrog točke $(0,0)$ ~ razvoj $\cos(t)$ okrog 0 za $t = xy^2$

$$\cos(xy^2) = 1 - \frac{x^2y^4}{2!} + \frac{x^4y^8}{4!} - \frac{x^6y^{12}}{6!} + \dots = \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k \cdot x^{2k} y^{4k}$$

$$x \cdot \cos(xy^2) = x - \frac{x^3y^4}{2!} + \frac{x^5y^8}{4!} - \frac{x^7y^{12}}{6!} + \dots = \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k x^{2k+1} y^{4k} =$$

$$= x - \frac{1}{2!} x^3y^4 + \frac{1}{4!} x^5y^8 - \frac{1}{6!} x^7y^{12} \pm \dots$$

b) koeficient pred $x^n y^m$: $\frac{1}{(n+m)!} \cdot \binom{n+m}{n} \cdot \frac{\partial^{n+m} f}{\partial x^n \partial y^m}(0,0)$

(to pomeni:

$$c_{ij} = \frac{1}{(i+j)!} \binom{i+j}{i} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0) = \frac{1}{i!\,j!} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0)$$)

• $\frac{\partial^7 f}{\partial x^3 \partial y^4}$ v Taylorjevi vrsti se pojavi pri

$$\frac{1}{7!}\binom{7}{3}\frac{\partial^7 f}{\partial x^3\partial y^4}(0,0)\cdot x^3y^4 = -\frac{x^3y^4}{2!}$$

$$\frac{\cancel{1}}{\cancel{7!}}\cdot\frac{\cancel{7!}}{3!\cdot 4!}\cdot\frac{\partial^7 f}{\partial x^3\partial y^4}(0,0) = -\frac{1}{2}$$

$$\frac{\partial^7 f}{\partial x^3\partial y^4}(0,0) = -\frac{1}{2}\cdot 3!\cdot 4! = -72$$

• $\frac{\partial^{61} f}{\partial x^{21}\partial y^{40}}$:

$(k=10)$

$$\frac{1}{(20)!}\cdot(-1)^{10}\cdot x^{21}\cdot y^{40} = \frac{1}{(61)!}\cdot\binom{61}{21}\cdot\frac{\partial^{61} f}{\partial x^{21}\partial y^{40}}(0,0)\cdot x^{21}y^{40}$$

$$\frac{\cancel{1}}{\cancel{(20)!}} = \frac{\cancel{1}}{\cancel{(61)!}}\cdot\frac{\cancel{(61)!}}{\cancel{(21)!}\cdot(40)!}\cdot\frac{\partial^{61} f}{\partial x^{21}\partial y^{40}}(0,0)$$

$$\frac{\partial^{61} f}{\partial x^{21}\partial y^{40}}(0,0) = 21\cdot(40)!$$

• $\frac{\partial^{62} f}{\partial x^{22}\partial y^{40}}(0,0)$: v Taylorjevi vrsti pri $\frac{1}{(62)!}\cdot\binom{62}{22}\cdot\frac{\partial^{62} f}{\partial x^{22}\partial y^{40}}(0,0)\cdot x^{22}y^{40}$

Ali obstaja tak člen v vsoti $\sum_{k=0}^{\infty}\frac{1}{(2k)!}(-1)^k x^{2k+1}y^{4k}$?

$22 = 2k+1$
$40 = 4k$ // NE

$$\Rightarrow \frac{\partial^{62} f}{\partial x^{22}\partial y^{40}}(0,0) = 0$$