# LINEARNA ALGEBRA 2020/21
# 2. VAJE: 12. 10. 2020

1. Pokaži, da sta vektorja $(x_1, x_2)$ in $(y_1, y_2)$ linearno odvisna natanko tedaj, ko je $x_1y_2 - y_1x_2 = 0$.

2. Izračunaj skalarna produkta:

   a. $\langle \vec{\imath} - \vec{\jmath} + \vec{k}, 2\vec{\imath} + 3\vec{\jmath} - \vec{k} \rangle$

   b. $\langle 2\vec{\imath} + \vec{k}, 2\vec{\jmath} - \vec{k} \rangle$

3. Naj bo $\vec{a} = (1, -3, 1), \vec{b} = (2, -3, 0)$ in $\vec{c} = (-1, 2, 1)$. Določi $\lambda$ tako, da bosta $\vec{a} + \lambda\vec{b}$ in $\vec{c}$ pravokotna. Določi $\mu$, da bo imel $\mu\vec{c}$ dolžino 1.

4. Določi kot med $\vec{a}$ in $\vec{b}$, če velja $\vec{a} + 3\vec{b} \perp 7\vec{a} - 5\vec{b}$ in $\vec{a} - 4\vec{b} \perp 7\vec{a} - 2\vec{b}$.

5. V trikotniku, ki ga določata vektorja $\vec{a}$ in $\vec{b}$ izrazi nožišče višine nad stranico, ki jo določa $\vec{a}$.

6. Določi kot med ploskvama v pravilnem tetraedru.

7. Za $\vec{a} = (1, 2, -1), \vec{b} = (1, 3, 0)$ in $\vec{c} = (2, 0, 1)$ izračunaj $(\vec{a} \times \vec{b}) \times \vec{c}$ in $\vec{a} \times (\vec{b} \times \vec{c})$.

$A \Leftrightarrow B$

Recimo, da je A res. Želimo pokazati da je B res.
Recimo, da B ni res. — || — A ni res.

$x = (x_1, x_2)$ $y = (y_1, y_2)$

$x, y$ linearno odvisna $\Rightarrow x_1y_2 - x_2y_1 = 0$

$x = \lambda y$ $\quad x_1 = \lambda y_1 \quad x_2 = \lambda y_2 \Rightarrow \lambda y_1 y_2 - \lambda y_2 y_1 = 0$

$x, y = 0$ $\quad \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$

$x_1y_2 - x_2y_1 = 0 \Rightarrow x, y$ linearno odvisna

$\boxed{x_1 = \lambda y_1}$ $\quad \lambda = \frac{x_2}{y_2} \quad x_1y_2 = x_2y_1 \; /:y_2 \rightarrow x_1 = \left(\frac{x_2}{y_2}\right) y_1$

$y_2 \neq 0$

$$x_2 \doteq \frac{x_2}{y_2} \cdot y_2 = x_2 \qquad x = \lambda y.$$

2. Izračunaj skalarna produkta:

   a. $\langle \vec{\imath} - \vec{\jmath} + \vec{k}, 2\vec{\imath} + 3\vec{\jmath} - \vec{k} \rangle$ $= \langle \vec{i}, 2\vec{i}+3\vec{j}-\vec{k} \rangle - \langle \vec{j}, 2\vec{i}+3\vec{j}-\vec{k} \rangle + \langle \vec{k}, 2\vec{i}+3\vec{j}-\vec{k} \rangle$

   b. $\langle 2\vec{\imath} + \vec{k}, 2\vec{\jmath} - \vec{k} \rangle$ $= 2\cdot 0 + 0\cdot 2 + 1(-1) = -1 \quad - \quad 3 \quad - 1 = -2$

$\mathbb{R}^3 \quad \vec{i} = (1,0,0), \vec{j} = (0,1,0), \vec{k} = (0,0,1)$ – standardna baza

$\langle \vec{i}, \vec{i} \rangle = \langle \vec{j}, \vec{j} \rangle = \langle \vec{k}, \vec{k} \rangle = 1$

$\langle \vec{i}, \vec{j} \rangle = \langle \vec{i}, \vec{k} \rangle = \langle \vec{j}, \vec{k} \rangle = 0$

$\langle \vec{v}+\vec{u}, \vec{t} \rangle = \langle \vec{v}, \vec{t} \rangle + \langle \vec{u}, \vec{t} \rangle.$

$\langle \vec{v}, \vec{u} \rangle = \langle \vec{u}, \vec{v} \rangle$

$\langle \lambda\vec{v}, \vec{u} \rangle = \lambda \langle \vec{v}, \vec{u} \rangle$

$\langle \vec{v}, \vec{v} \rangle = \|\vec{v}\|^2 \geq 0$ in $=$, če je $\vec{v} = 0$.

$$\langle \vec{i}, 2\vec{i}+3\vec{j}-\vec{k} \rangle = 2$$
$$\|$$
$$2\underbrace{\langle \vec{i}, \vec{i} \rangle}_{1} + \underbrace{\langle \vec{i}, 3\vec{j} \rangle}_{0} + \underbrace{\langle \vec{i}, -\vec{k} \rangle}_{0}$$

$\langle \vec{v}, \vec{u} \rangle$

$$\frac{\langle \vec{v}, \vec{u} \rangle \cdot \vec{u}}{\langle \vec{u}, \vec{u} \rangle} = \cancel{\frac{\vec{v}\vec{u}\cdot\vec{u}}{\vec{u}\vec{u}}} \cancel{\neq} \vec{V}$$

3. Naj bo $\vec{a} = (1, -3, 1), \vec{b} = (2, -3, 0)$ in $\vec{c} = (-1, 2, 1)$. Določi $\lambda$ tako, da bosta $\vec{a} + \lambda\vec{b}$ in $\vec{c}$ pravokotna. Določi $\mu$, da bo imel $\mu\vec{c}$ dolžino 1.

$\langle \vec{a}+\lambda\vec{b}, \vec{c} \rangle = 0$

$\|$

$\langle \vec{a}, \vec{c} \rangle + \lambda \langle \vec{b}, \vec{c} \rangle = 0 \rightarrow -\langle a, \vec{c} \rangle = \lambda \langle \vec{b}, \vec{c} \rangle$

$\langle (1,-3,1), (-1,2,1) \rangle = -1 - 6 + 1 = -6$

$\langle (2,-3,0), (-1,2,1) \rangle = -2 - 6 + 0 = -8$

$6 = -\lambda \cdot 8 \rightarrow -\frac{3}{4} = \lambda$

$\langle (-1,2,1), (-1,2,1) \rangle = 1+4+1$

$\|\vec{V}\| = 1 \Rightarrow \|-\vec{V}\| = 1$

$1 = \|\mu\vec{c}\|^2 = \langle \mu\vec{c}, \mu\vec{c} \rangle = \mu^2 \langle \vec{c}, \vec{c} \rangle = \mu^2 6$

$\|\mu\vec{c}\| = |\mu| \|\vec{c}\|$ $\quad \mu^2 = \frac{1}{6} \Rightarrow \mu = \pm\sqrt{\frac{1}{6}}$

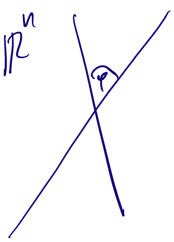
$$\vec{v} \rightsquigarrow \frac{\vec{v}}{\sqrt{\langle \vec{v}, \vec{v} \rangle}} = \frac{\vec{v}}{\|\vec{v}\|} \text{ – normiran vektor}$$

4. Določi kot med $\vec{a}$ in $\vec{b}$, če velja $\vec{a} + 3\vec{b} \perp 7\vec{a} - 5\vec{b}$ in $\vec{a} - 4\vec{b} \perp 7\vec{a} - 2\vec{b}$.

$$\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \|\vec{b}\| \cdot \cos\varphi$$

$\varphi$ – kot med $\vec{a}, \vec{b}$

$\mathbb{R}^n$



$$\frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\| \cdot \|\vec{b}\|}$$

$$\langle \vec{a} + 3\vec{b}, 7\vec{a} - 5\vec{b} \rangle = 0$$

$$7\langle \vec{a}, \vec{a} \rangle - 5\langle \vec{a}, \vec{b} \rangle + 21\langle \vec{b}, \vec{a} \rangle - 15\langle \vec{b}, \vec{b} \rangle = 0$$

$$7\langle \vec{a}, \vec{a} \rangle + 16\langle \vec{a}, \vec{b} \rangle - 15\langle \vec{b}, \vec{b} \rangle = 0$$

$$\langle \vec{a} - 4\vec{b}, 7\vec{a} - 2\vec{b} \rangle = 0 \qquad -1$$

$$7\langle \vec{a}, \vec{a} \rangle - 30\langle \vec{a}, \vec{b} \rangle + 8\langle \vec{b}, \vec{b} \rangle = 0$$

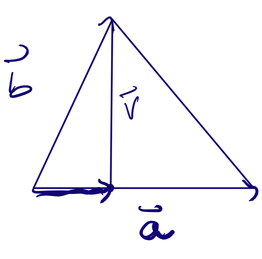
$$46\langle \vec{a}, \vec{b} \rangle - 23\langle \vec{b}, \vec{b} \rangle = 0 \quad / :23 \qquad 2\langle a, b \rangle - \langle b, \vec{b} \rangle = 0$$

$$2\langle \vec{a}, \vec{b} \rangle = \langle \vec{b}, \vec{b} \rangle \qquad 7\langle \vec{a}, \vec{a} \rangle + 8\langle \vec{b}, \vec{b} \rangle - 15\langle \vec{b}, \vec{b} \rangle = 0$$

$$7\langle \vec{a}, \vec{a} \rangle = 7\langle \vec{b}, \vec{b} \rangle \Rightarrow \langle \vec{a}, \vec{a} \rangle = \langle \vec{b}, \vec{b} \rangle$$

$$\|a\|^2 = \|b\|^2 = \|a\|\|b\| = \langle b, b \rangle \qquad \frac{\langle \vec{a}, \vec{b} \rangle}{\langle b, b \rangle} = \frac{\langle a, b \rangle}{2\langle a, b \rangle} = \frac{1}{2} \qquad \varphi = \pm\frac{1}{3}\pi + 2\pi k$$

5. V trikotniku, ki ga določata vektorja $\vec{a}$ in $\vec{b}$ izrazi nožišče višine nad stranico, ki jo določa $\vec{a}$.



$$\vec{v} \perp \vec{a} = 0$$

$$\vec{v} = -\lambda\vec{a} + \vec{b}$$

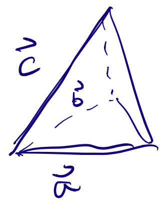
$$\langle -\lambda\vec{a} + \vec{b}, \vec{a} \rangle = 0 \qquad -\lambda\langle \vec{a}, \vec{a} \rangle + \langle a, b \rangle = 0$$

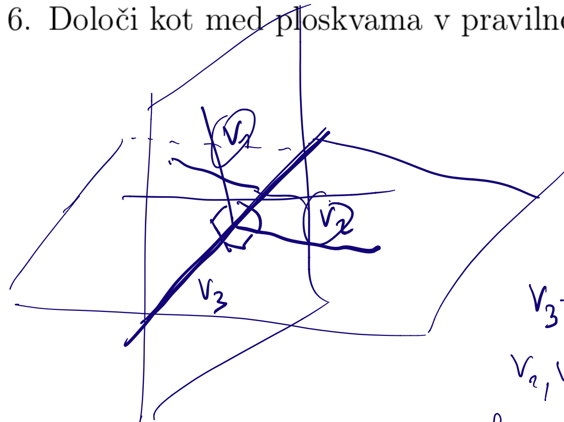
$$\lambda\langle \vec{a}, \vec{a} \rangle = \langle \vec{a}, \vec{b} \rangle \qquad \lambda = \frac{\langle \vec{a}, \vec{b} \rangle}{\langle \vec{a}, \vec{a} \rangle}$$

$$\lambda\vec{a} = \frac{\langle \vec{a}, \vec{b} \rangle}{\langle \vec{a}, \vec{a} \rangle} \cdot \vec{a} \quad \text{pravokotna projekcija } \vec{b} \text{ na } \vec{a}$$

$$\frac{\langle \vec a, \vec b\rangle}{\langle a, \vec a\rangle}\cdot \vec a \qquad \vec v = b - \frac{\langle \vec a, \vec b\rangle}{\langle a, \vec a\rangle}\cdot \vec a$$

6. Določi kot med ploskvama v pravilnem tetraedru.




max { $\varphi$ | $\varphi$ kot med premicama na ploskvah }

$v_3$ = presek ploskev

$v_1, v_2 \perp v_3$

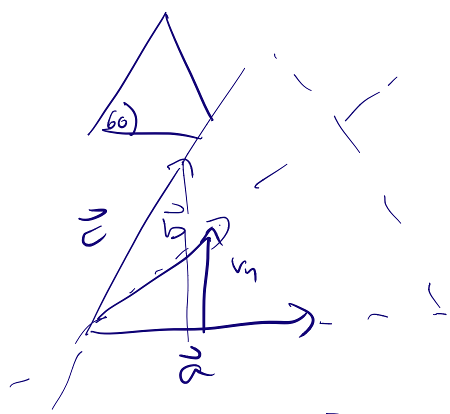
$\varphi = \angle v_1, v_2$

$$\|\vec a\| = \|\vec b\| = \|\vec c\| = 1$$

$$\langle \vec a, \vec b\rangle = \frac{1}{2}$$

$$\langle \vec b, \vec c\rangle = \frac{1}{2}$$

$$\langle \vec a, \vec c\rangle = \frac{1}{2}$$




$\vec a$ - v preseku

$$\vec v_1 = \vec b - \tfrac{1}{2}\vec a$$

$$\vec v_2 = \vec c - \tfrac{1}{2}\vec a$$

$$\langle \vec v_1, \vec v_1\rangle = \langle \vec b - \tfrac{1}{2}\vec a, \vec b - \tfrac{1}{2}\vec a\rangle = \underline{\langle \vec b, \vec b\rangle} - 1\underline{\langle \vec a, \vec b\rangle} + \frac{1}{4}\underline{\langle \vec a, \vec a\rangle}$$

$$= 1 - \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$\vec b \to \vec c$

$$\langle \vec v_2, \vec v_2\rangle = \frac{3}{4}$$

$$\langle \vec v_1, \vec v_2\rangle = \langle \vec b - \tfrac{1}{2}\vec a, \vec c - \tfrac{1}{2}\vec a\rangle$$

$$= \langle \vec b, \vec c\rangle - \frac{1}{2}\langle \vec b, \vec a\rangle - \frac{1}{2}\langle a, \vec c\rangle + \frac{1}{4}\langle \vec a, \vec a\rangle$$

$$= \frac{1}{2} - \frac{1}{4} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} = \frac{1}{4}$$

$$\cos\varphi = \frac{\langle \vec v_1, \vec v_2\rangle}{\|v_1\|\cdot\|v_2\|} = \frac{\frac{1}{4}}{\sqrt{\frac{3}{4}\cdot\frac{3}{4}}} = \frac{1}{3}$$

$$\underline{\varphi = \arccos\left(\tfrac{1}{3}\right)}$$

$\mathbb{R}^3$ Vektorski produkt

$\vec{v}, \vec{u}$ $\quad$ $\vec{v}\times\vec{u} \perp \vec{v}, \vec{u}$ $\quad$ $|\vec{v}\times\vec{u}| = \square$ paralelogram določen z $\vec{v}, \vec{u}$

$\vec{v} = (v_1, v_2, v_3)$

$\vec{u} = (u_1, u_2, u_3)$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} & i & j \\ v_1 & v_2 & v_3 & v_1 & v_2 \\ u_1 & u_2 & u_3 & u_1 & u_2 \end{vmatrix}$$

smer določena z desno sučnim vijakom

$\vec{v}\times\vec{u} = (v_2u_3 - v_3u_2,\ v_3u_1 - v_1u_3,\ v_1u_2 - v_2u_1)$

1. naloga $\quad x\times y = (0,0,*)$

$x \mapsto (x_1, x_2, 0)$ $\quad \|x\times y\| = 0$

$y \mapsto (y_1, y_2, 0)$

7. Za $\vec{a} = (1, 2, -1), \vec{b} = (1, 3, 0)$ in $\vec{c} = (2, 0, 1)$ izračunaj $(\vec{a}\times\vec{b})\times\vec{c}$ in $\vec{a}\times(\vec{b}\times\vec{c})$.

$(a \circledast b)\circledast c = a \circledast (b \circledast c)$

asociativnost

$(1,2,-1)$

$(1,3,0)$

$\vec{a}\times\vec{b} = (2\cdot 0 - (-1)3, -1, 3-2)$

$= (3,-1,1)$

$(3,-1,1)$

$(2,0,1)$

$(\vec{a}\times\vec{b})\times\vec{c} = (-1, 2-3, 2)$

$= (-1,-1,2)$

$(1,3,0)$

$(2,0,1)$

$\vec{b}\times\vec{c} = (3,-1,-6)$

$(1,2,-1)$

$(3,-1,-6)$

$\vec{a}\times(\vec{b}\times\vec{c}) = (-13, 3, -7)$

v splošnem

$(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

1. Vsoto $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}$ izrazi z enim vektorskim produktom.

$\boxed{\vec{a}\times\vec{a} = \vec{b}\times\vec{b} = \vec{c}\times\vec{c} = 0}$

$(\vec{x}+\vec{y})\times\vec{a} = (\vec{x}\times\vec{a}) + (\vec{y}\times\vec{a})$

$\vec{a}\times\vec{b} - \vec{a}\times\vec{c} + \vec{b}\times\vec{c} = \vec{a}\times(\vec{b}-\vec{c}) + \vec{b}\times\vec{c} - \vec{b}\times\vec{b} \longrightarrow \vec{b}\times(\vec{c}-\vec{b})$

$\vec{b}\times\vec{c} - \vec{c}\times\vec{c} \longrightarrow (\vec{b}-\vec{c})\times\vec{c}$

$-(\vec{b}-\vec{c})$

$\vec{a}\times(\vec{b}-\vec{c}) + \vec{b}\times(\vec{c}-\vec{b})$

$(\vec{a}-\vec{b})\times(\vec{b}-\vec{c})$

$\vec{a}\times(\vec{b}-\vec{c}) + (\vec{b}-\vec{c})\times\vec{c}$

$= a\times(\vec{b}-\vec{c}) - \vec{c}\times(\vec{b}-\vec{c})$