ANALIZA 1 – IŠRM vaje

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- ZAČETEK OB 8.30

NARAVNA ŠTEVILA

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

- seštevanje, množenje ✓
- deljenje, odštevanje $\not\checkmark$ (znotraj $\mathbb{N}$)

NARAVNA / POPOLNA INDUKCIJA

- lastnost P
- $P(0)$, $\forall n \in \mathbb{N}$. $\underline{P(n) \Rightarrow P(n+1)}$ } $*$

↳ "0 izpolnjuje lastnost P"

"če neka lastnost velja za n, potem velja tudi za $n+1$"

$\cancel{x} \Rightarrow \forall n \in \mathbb{N}.\ P(n)$

"P velja za vsa naravna števila"

Dokaži z indukcijo za vsa naravna št. $n \geq 0$:

- $$\sum_{k=0}^{n} k := 0 + 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$n = 0$

$0 = 0$ ✓

$n \Rightarrow n+1$

$$\left.\sum_{k=0}^{n} k = \frac{n(n+1)}{2}\right\} \text{IH}$$

$$\sum_{k=0}^{n+1} k = \sum_{k=0}^{n} k + (n+1) \overset{I+I}{=}$$

$$= \frac{n(n+1)}{2} + (n+1) =$$

$$= \frac{n(n+1) + 2(n+1)}{2} =$$

$$= \underline{\underline{\frac{(n+1)(n+2)}{2}}}$$

$$\sum_{k=0}^{n} k^2 = 0^2 + 1^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$6 \mid n^3 - n \quad \text{za} \quad \forall n \in \mathbb{N}$$

$$\hookrightarrow \quad n^3 - n = 6 \cdot k, \quad k \in \mathbb{N}$$

$\underline{n = 0}$

$6 \mid 0$?

$0 = 6 \cdot 0$ ✓

$\underline{n \Rightarrow n+1}$

$$IH: \quad 6 \mid n^3 - n \quad \approx \quad n^3 - n = 6 \cdot k$$

$$6 \mid (n+1)^3 - (n+1)$$

$$(n+1)^3 - (n+1) =$$

$$= n^3 + 3n^2 + 3n + 1 - n - 1 =$$

$$= (n^3 - n) + 3n^2 + 3n \overset{IH}{=}$$

$$= 6 \cdot k + \underbrace{3(n^2 + n)}$$

$$6 = 2 \cdot 3$$

$$\gcd(2,3) = 1$$

$$\Downarrow$$

$$6 \mid m \iff 2 \mid m$$

in

$$3 \mid m$$

$$= 6 \cdot k + \underbrace{3n(n+1)}_{6 \mid 3n(n+1)}$$

$\checkmark$

ker $\quad 2 \mid n(n+1)$

$$n^3 - n = n(n^2 - 1) =$$

$$= (n-1)n(n+1)$$

$$n! \leq 2 \cdot \left(\frac{n}{2}\right)^n \quad ; \quad n \geq 1$$

$$n = 1$$

$$1! \leq 2 \cdot \left(\frac{1}{2}\right)^1$$

$$1 \leq 1 \quad \checkmark$$

$$n \Rightarrow n+1$$

$$\text{IH}: \quad n! \leq 2 \cdot \left(\frac{n}{2}\right)^n$$

$$(n+1)! = n!\,(n+1) \overset{\text{IH}}{\leq}$$

$$\leq 2 \cdot \left(\frac{n}{2}\right)^n (n+1)$$

$$\begin{array}{cc} \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} \\ \left(\frac{n+1}{2}\right)^n & \frac{n+1}{2} \end{array}$$

$$\leq 2 \cdot \left(\frac{n+1}{2}\right)^{n+1}$$

• $n \geq 1$

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2^{n-1}})$$

$$= 1 + x + x^2 + x^3 + \ldots + x^{2^n - 1}$$

• $\underline{n = 1}$

$$1 + x = 1 + x \checkmark$$

• $\underline{n \Rightarrow n+1}$

$$(1+x)(1+x^2)\ldots(1+x^{2^{n-1}})(1+x^{2^n}) \overset{IH}{=}$$

$$= (1 + x + x^2 + x^3 + \ldots + x^{2^n - 1})(1 + x^{2^n}) =$$

$$=$$

$$\overset{?}{=}$$

$$=$$

$$= 1 + x + x^2 + \ldots + x^{2^n - 1} + x^{2^n} + x^{2^n + 1} +$$

$$+ x^{2^n + 2} + \ldots + x^{2^n - 1 + 2^n} =$$

$\hookrightarrow \; 2^n - 1 + 2^n = 2 \cdot 2^n - 1 = 2^{n+1} - 1$

$= 1 + x + x^2 + x^3 + \dots + x^{2^{n+1}-1}$

CELA ŠTEVILA:

$\dots \; -2, \; -1, \; 0, \; 1, \; 2, \; \dots$

odštevanje: ✓

deljenje //

$f: \mathbb{N} \longrightarrow \mathbb{Z}$, bijekcija

Ideja: n sodo $\rightsquigarrow f(n) \geq 0$

n liho $\rightsquigarrow f(n) < 0$

$$f(n) = \begin{cases} \frac{n}{2} & ; \ n \text{ sodo} \\ -\frac{n+1}{2} & ; \ n \text{ liho} \end{cases}$$

$f(\mathbb{N}) = \mathbb{Z}$ ✓

f je injektivna ✓

RACIONALNA ŠTEVILA

$$\mathbb{Q} = \left\{ \frac{m}{n} \ ; \ m, n \in \mathbb{Z}, \ n \neq 0 \right\}$$

$x^2 - 2$, polinom v $\mathbb{Q}[x]$, nima rešitve v $\mathbb{Q}$

$\mathbb{Q} \subsetneq \mathbb{R}$

$|\mathbb{Q}| = |\mathbb{N}|$

$|\mathbb{N}| \lneq |\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$

- $|A| \lneq |\mathcal{P}(A)|$
- $\exists B$, $|\mathbb{N}| \lneq |B| \lneq |\mathcal{P}(\mathbb{N})|$?

(HIPOTEZA KONTINUUMA)

ZFC aksiomi

$\mathbb{Q}$ je zaprta za seštevanje, množ.

$a, b \in \mathbb{Q} \Rightarrow a + b \in \mathbb{Q}$

$a, b \in \mathbb{Q} \Rightarrow ab \in \mathbb{Q}$

1.

$a \in \mathbb{Q},\ b \notin \mathbb{Q} \Rightarrow a + b \in \mathbb{Q}$?

$a + b \in \mathbb{Q},\ -a \in \mathbb{Q} \Rightarrow b \in \mathbb{Q}$

Sklep: $a+b \notin \mathbb{Q}$

$a, b \neq 0,\ a \in \mathbb{Q},\ b \notin \mathbb{Q} \Rightarrow ab \notin \mathbb{Q}$

$n \in \mathbb{Q},\ n \geq 1,\ n^2 \in \mathbb{N} \Rightarrow n \in \mathbb{N}$

$n = \frac{a}{b}$; $a \perp b$; $a, b \in \mathbb{N},\ b \neq 0$

↳ tj. $\gcd(a, b) = 1$

Hoćemo: $b = 1$

$n^2 = \frac{a^2}{b^2} \in \mathbb{N}$

$n^2 b^2 = a^2$ ⤸ $\gcd(a, b) = 1$

$b^2 \mid a^2$ (A)

$\frac{a^2}{b^2}$ okrajšan ulomek

$\gcd(a, b) = 1 \Rightarrow \gcd(a^2, b^2) = 1$

Dokaz z razcepom na prafaktorje

$a = p_1^{k_1} \cdot \ldots \cdot p_m^{k_m}$

$b = q_1^{l_1} \cdot \ldots \cdot q_n^{l_n}$

$\Rightarrow \quad p_i \neq q_j \quad \forall i,j \quad (*)$

$a^2 = p_1^{2k_1} \cdot \ldots \cdot p_m^{2k_m}$

$b^2 = q_1^{2l_1} \cdot \ldots \cdot q_n^{2l_n}$

$(*) \Rightarrow \underline{\gcd(a^2, b^2) = 1}$

(B)

$b^2 = 1 \Rightarrow b = 1$

Torej je $n \in \mathbb{N}$.

Dokaži, da so naslednja števila iracionalna:

- $n \geq 1$, $\sqrt{n^2+n+1}$

$\hookrightarrow \notin \mathbb{Q}$

Iz prejšnje naloge:

$$\sqrt{n^2+n+1} \in \mathbb{Q} \Rightarrow \sqrt{n^2+n+1} \in \mathbb{N}$$

Dokazujemo: $\sqrt{n^2+n+1} \notin \mathbb{N}$

$$n^2 < n^2+n+1 < (n+1)^2$$

$\sqrt{\ }$ inj.

$$n < \sqrt{n^2+n+1} < n+1$$

$$\Rightarrow \sqrt{n^2+n+1} \notin \mathbb{N}$$

$$\Rightarrow \sqrt{n^2+n+1} \notin \mathbb{Q}$$

✓

$\sqrt{\frac{n}{n+1}}$ je iracionalno za $n \geq 1$
$n \in \mathbb{N}$

Dokaz s protislovjem:

$$\sqrt{\frac{n}{n+1}} = \frac{a}{b} \quad ; \quad a \perp b$$

$a, b \in \mathbb{N} \setminus \{0\}$

+ prejšnja naloga

Sledi:

$$\frac{n}{n+1} = \frac{a^2}{b^2} \quad ; \quad a^2 \perp b^2$$

$$n b^2 = (n+1) a^2$$

$$n b^2 = n a^2 + a^2$$

$$n (b^2 - a^2) = a^2$$

$$\hookrightarrow b^2 - a^2 \mid a^2$$

$$b^2 - a^2 = 1$$

$$\hookrightarrow b > a$$

$b - a \geq 1$

$\Rightarrow b + a > 1$

$$b^2 - a^2 = (b+a)(b-a) > 1$$

m, n ; $\quad m \perp n$

$m - n, n$ tuja ?

Pa recimo: nista tuja $\quad$ če $k \neq 1$

$k \mid m - n, \quad k \mid n \Rightarrow k \mid m$

$\cancel{\ }$ $k = 1$

Sklep: $\sqrt{\dfrac{n}{n+1}} \in \mathbb{Q}$

$$\left.\begin{aligned} m - n &= a \cdot k \\ n &= b \cdot k \end{aligned}\right] + \quad ; \quad a, b \in \mathbb{Z}$$

$m = (a + b) k$

Poišči dve iracionalni števili x, y, da je $x^y \in \mathbb{Q}$

$y = \sqrt{2}$, $x = \sqrt{3}$

$x^y = \sqrt{2}^{\sqrt{3}}$

a) $x^y \in \mathbb{Q}$ ✓

b) $x^y \notin \mathbb{Q}$

$(\sqrt{2})^{\sqrt{3}}$

$\left((\sqrt{2})^{\sqrt{3}}\right)^{\sqrt{3}} =$

$x^y = \sqrt{3}^{\sqrt{2}}$

a) $x^y \in \mathbb{Q}$ ✓

b) $x^y \notin \mathbb{Q}$

$(\sqrt{3})^{\sqrt{2}} \to \notin \mathbb{Q}$

$\mathbb{Q}$ ✓

$\left(\underbrace{(\sqrt{3})^{\sqrt{2}}}\right)^{\sqrt{2}} = \sqrt{3}^{2} = \underline{\underline{3}}$

b, $\sqrt{3}^{\sqrt{2}} \notin \mathbb{Q}$

Realna števila

→ algebraična $\mathbb{A}$

$|\mathbb{A}| = |\mathbb{N}|$

→ transcendentna

↳ sešt., odšt.

↳ vsak polinom lihe stopnje v $\mathbb{R}[x]$ ima rešitev v $\mathbb{R}$

AKSIOM O KONTINUUMU (AK)

$\emptyset \neq A \subseteq \mathbb{R}$

obstaja

$\exists M \in \mathbb{R}, \quad \forall x \in A. \ x \leq M$

$\overset{AK}{\Longrightarrow}$ A ima v $\mathbb{R}$ najmanjšo zg. mejo (supremum)

$\sup (a, b) = b \notin (a, b)$

$\sup [a, b] = b \in [a, b]$

↳ b je maksimum od A

- $[a, b] \subseteq \mathbb{R}$
- $\{(a_i, b_i) \subseteq \mathbb{R},\ i \in I\}$, $\overset{\text{recimo}}{\longrightarrow} \mathbb{N}$

$$[a, b] \subseteq \bigcup_{i \in I} (a_i, b_i) \quad (*)$$

$i_1, i_2, i_3, \dots, i_n \in I$:

$$[a, b] \subseteq (a_{i_1}, b_{i_1}) \cup \dots \cup (a_{i_n}, b_{i_n})$$

Namig: AK na neki množici

$A := \{c \in [a, b] \mid$ obstaja končno podpokritje $\{(a_i, b_i)\}$ za $[a, c]\}$
$\subseteq [a, b]$

Ideja: a) dokaz, da ima A supremum

b) $\sup A = b$

c) $b \in A$

a) A neprazna in navzg. omejena

$a \in A$

$$[a,a] := \{x \in \mathbb{R} \mid a \leq x \leq a\} = \{a\}$$

$$(*) \Rightarrow a \in \bigcup_{i \in I} (a_i, b_i)$$

$$\Rightarrow \exists (a_{i_1}, b_{i_1}) : a \in (a_{i_1}, b_{i_1})$$ ✓

b je zgornja meja za $A \subseteq [a,b]$

AK $\Rightarrow$ A ima supremum v $\mathbb{R}$

$\sup A \leq b$, mi želimo $\sup A = b$

Pa recimo:

$c := \sup A < b$

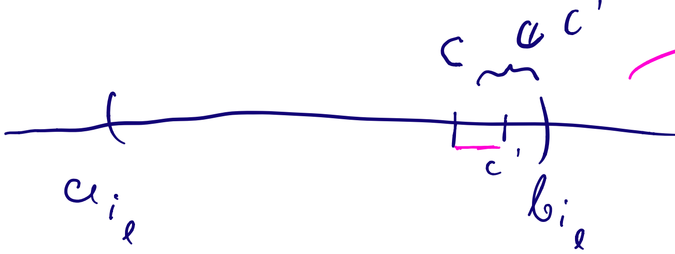
$[a,c]$ največji interval, za katerega obstaja končno podpokritje

$$[a,c] \subseteq \underbrace{(a_{i_1}, b_{i_1}) \cup (a_{i_2}, b_{i_2}) \cup \dots \cup (a_{i_k}, b_{i_k})}_{\text{unija}}$$

obstaja $c' > c$:

$$[a,c'] \subseteq (a_{i_1}, b_{i_1}) \cup \dots \cup (a_{i_k}, b_{i_k})$$

$$\Rightarrow c \in (a_{i_\ell}, b_{i_\ell}) \quad ; \quad 1 \le \ell \le k$$



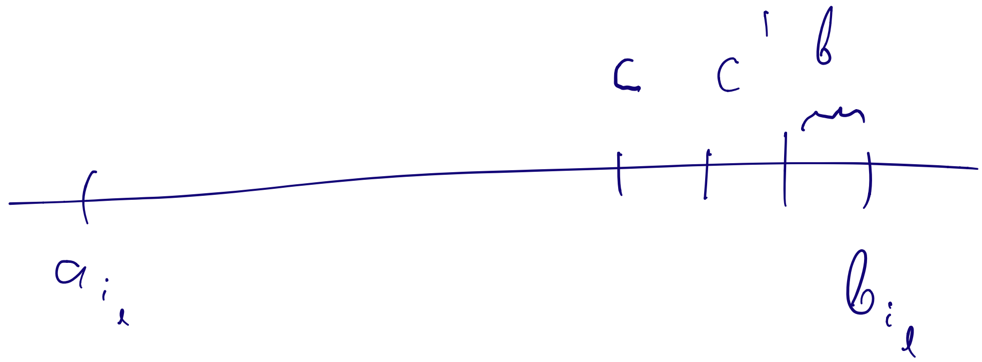
$\to [c,c'] \subseteq (a_{i_\ell}, b_{i_\ell})$

$\subseteq$ unija

$$[a,c'] \subseteq (a_{i_1}, b_{i_1}) \cup \dots \cup (a_{i_k}, b_{i_k})$$

Našli smo $c' > c$, da je $[a,c'] \subseteq$ unija

Ali lahko privzamemo: $c' \in A$? ✓



Sklep: c ni najm. zg. meja za A

$\Rightarrow b = \sup A$

$b \in A$.

(s protislovjem)