Poišči supremum, maksimum, minimum in infimum množice, če obstaja.

$$A = \left\{ 1 + \frac{1}{n} \;\middle|\; n \geq 1 \right\}$$

$$= \left\{ 2, \frac{3}{2}, \frac{4}{3}, \ldots, \frac{n+1}{n}, \ldots \right\}$$

$2 = \sup A$

$\forall x \in A .\; x \leq 2$

Hkrati: $2 \in A$

$2 = \max A = \sup A$ ✓

$1 = \inf A$

$\forall x \in A .\; 1 + \frac{1}{n} \geq 1$ $\leftarrow$ 1 je spodnja meja

1 je največja sp. meja

$c > 0$

$1 + c > 1 + \frac{1}{n}$ za nek $n \in \mathbb{N}$

$\Updownarrow$

$n > \frac{1}{c}$ — '' —

arhimedskost naravnih števil

Lahko tudi:

$$1 = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

Minimum ni dosežen, saj $1 \notin A$.

$$B = \left\{ \frac{n-\sqrt{n}}{n+1} \,;\, n \in \mathbb{N} \right\} =$$

$$= \left\{ 0, \frac{2-\sqrt{2}}{3}, \frac{3-\sqrt{3}}{4}, \frac{4-2}{5}, \dots, \frac{100-10}{101}, \dots, \dots, \frac{10000-100}{10001}, \dots \right\}$$

$$\sup B = 1$$

$$\frac{n-\sqrt{n}}{n+1} \le \frac{n}{n+1} < 1 \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad \sup B \le 1$$

Recimo: $c < 1$, tak, da je

$$c \geq \frac{n-\sqrt{n}}{n+1} \quad \text{za vse } n \in \mathbb{N}$$

$$c(n+1) \geq n - \sqrt{n}$$

$$cn + c \geq n - \sqrt{n}$$

$$\Rightarrow c \geq n - cn - \sqrt{n} = \underbrace{\sqrt{n}}_{\to\infty}\overbrace{\left(\overbrace{\sqrt{n}\,(\underbrace{1-c}_{>0})}^{\infty} - 1\right)}^{\infty}$$

$\sqrt{n}\left(\sqrt{n}\underbrace{(1-c)}_{>0}-1\right) > c$ za dovolj velike $n \in \mathbb{N}$

Torej ne obstaja $c < 1$, da je $c \geq \frac{n-\sqrt{n}}{n+1}$

$\Rightarrow \sup B = 1$

$1 \notin B \Rightarrow \max B$ ne obstaja

$0 = \inf B = \min B$

$\hookrightarrow 0 \in B$

$C = \{x \in (0,1) \mid x$ ima v dec. zapisu vsaj eno enico$\}$

$\sup C = 1$, $-$ $\max C$ ne obstaja

$y \in (0,1)$

$y = 0.y_1 y_2 y_3 y_4 y_5 \ldots$

$y_i < 9 \quad (y \neq 1)$ $\quad x_i = y_i + 1$

$x = 0.y_1 y_2 \ldots y_{i-1} x_i 1 0 \ldots$

Enako: $\inf C = 0$, $\min C$ ne obstaja

A, kdaj obstaja $\max A$?

↳ prvi nujni pogoj: A je neprazna in navzgor omejena

$\sup A$

↳ $\sup A \in A$

$x \in \sup A \iff \forall y \in A: y \leq x$

↳ $\sup A \notin A$

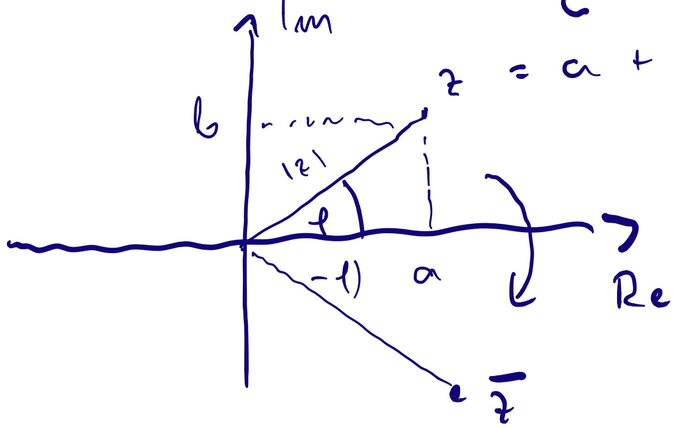
$x = \max A$, torej je v A. Ker ni supremum,

$x \in \max A \iff x \in A$ in $\forall y \in A: y \leq x$

obstaja $y \in A$; $x < y$

Kompleksna števila

$\mathbb{C}$



$z = a + ib = |z|(\cos\varphi + i\sin\varphi)$

$\varphi = \arg z$

Konjugiranje: $z = a + ib$

$\bar{z} = a - ib$

Nariši kompleksne množice v ravnini:

- $A = \{ z \in \mathbb{C} \mid z = \overline{-iz} \}$

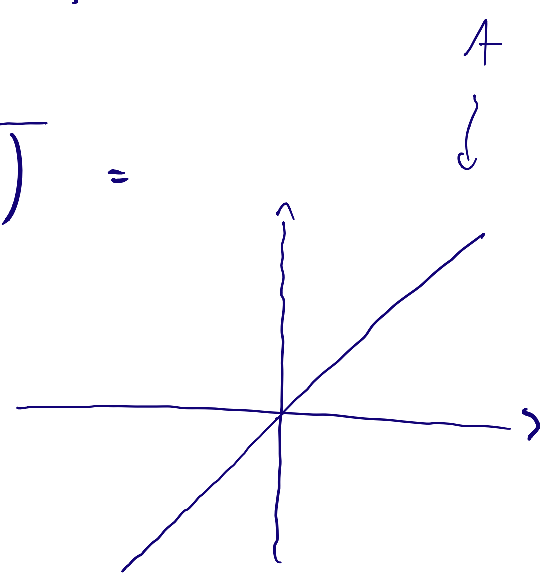
$z = a + ib$

$a + ib = \overline{-i(a+ib)} =$

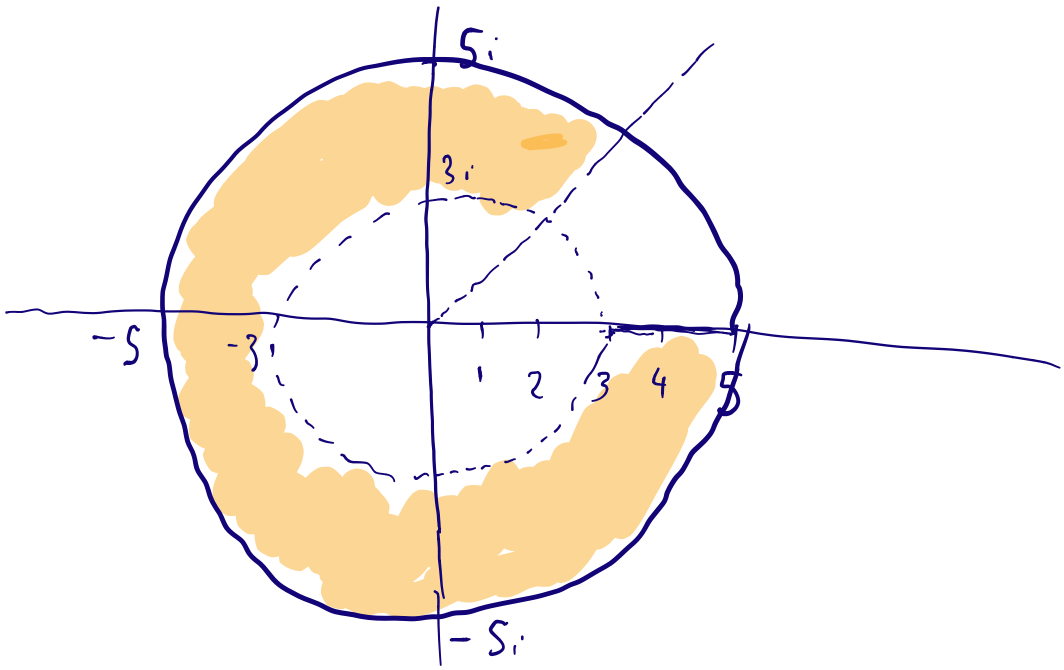
$= i(a - ib)$

$= ia + b$

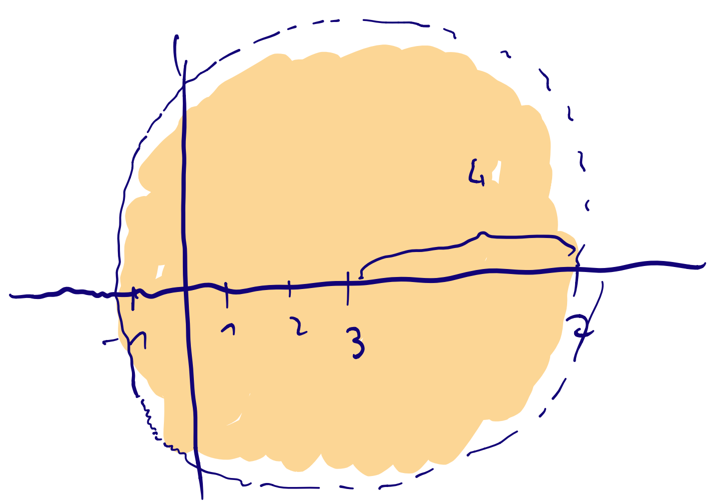
$\Rightarrow \quad a = b$



- $B = \left\{ z \in \mathbb{C} \;\middle|\; \frac{\pi}{4} < \arg(z) \leq 2\pi, \right.$
$\left. 3 < |z| \leq 5 \right\}$



- $C = \{ z \in \mathbb{C} \mid |z-3| < 4 \}$



Naj bo $z \in \mathbb{C}$, $|z| = 1$, $z \neq 1$

Dokaži: $i\,\frac{z+1}{z-1} \in \mathbb{R}$

$z = a + ib$, $a^2 + b^2 = 1$, $b \neq 0$ (doma)

Drug način

$$i\,\frac{z+1}{z-1} = \overline{i\,\frac{z+1}{z-1}}$$

$$i\,\frac{z+1}{z-1} = \overline{i\,\frac{z+1}{z-1}}$$

$$i\,\frac{z+1}{z-1} = -i\,\frac{\bar{z}+1}{\bar{z}-1} \quad / : i \quad / \cdot (z-1)(\bar{z}-1)$$

$z \neq 1$

$$(z+1)(\bar{z}-1) = (\bar{z}+1)(z-1)$$

$$z\bar{z} - z + \bar{z} - 1 = -z\bar{z} + \bar{z} - z + 1$$

$$z\bar{z} - 1 = -z\bar{z} + 1$$

$$|z|^2 = z\bar{z} = 1$$

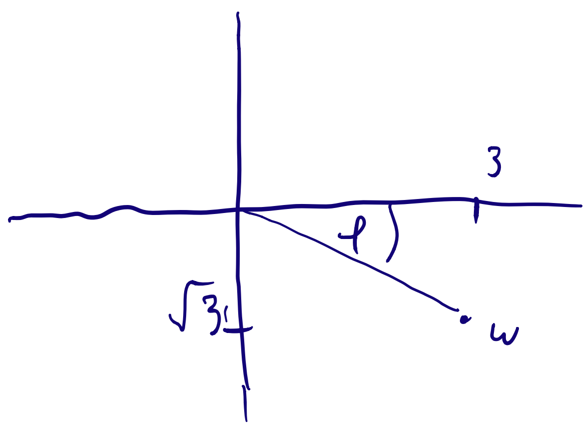
Reši kompleksno enačbo

- $z^3 = \underbrace{3 - i\sqrt{3}}_{w}$

$|w| = \sqrt{9+3} = \sqrt{12}$

$\arg w = \varphi = \frac{11\pi}{6}$

$\tan \varphi = -\frac{\sqrt{3}}{3}$



$\varphi = -\frac{\pi}{6} = \frac{11\pi}{6}$

$z^3 = \sqrt{12}\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right)$

$\hookrightarrow = |z|^3\left(\cos(3\varphi) + i\sin(3\varphi)\right)$

$\hookrightarrow z = |z| (\cos\psi + i\sin\psi)$

$\hookrightarrow \arg z = \psi$

$|z| = \sqrt[3]{\sqrt{12}} = \sqrt[6]{12}$

$$3\psi = \frac{11\pi}{6} + 2k\pi \qquad |:3$$

$\hookrightarrow k \in \mathbb{Z}$

$$\psi = \frac{11\pi}{18} + \frac{2k\pi}{3} \qquad k = 0, 1, 2$$

- $n \geq 1, n \in \mathbb{N}$

$z^n = \bar{z}$

$z = |z| (\cos\varphi + i\sin\varphi)$

$z^n = |z|^n (\cos(n\varphi) + i \cdot \sin(n\varphi))$

$\bar{z} = |z| (\cos(-\varphi) + i \cdot \sin(-\varphi))$

$\bar{z} = |z| (\underbrace{\cos\varphi}_{=\cos(-\varphi)} - \underbrace{i \cdot \sin\varphi}) =$

$\hookrightarrow -\sin(-\varphi)$

$$|z|^n\left(\cos(n\varphi) + i\sin(n\varphi)\right) =$$
$$= |z|\left(\cos(-\varphi) + i\sin(-\varphi)\right)$$

Abs. vrednosti:

$|z|^n = |z|$;

a) $n = 1$

Abs. vrednost ni določena

b) $n \geq 2$

$|z| \in \{0, 1\}$

$|z|^n = |z|$

Imamo realen polinom

$x^n - x = x(x^{n-1} - 1) =$

$= x(x-1)(1 + x + \dots + x^{n-2})$

ni rešljivo za $x > 0$

Argument

$n\varphi = -\varphi + 2k\pi$; $k \in \mathbb{Z}$

$(n+1)\varphi = 2k\pi$

$\varphi = \frac{2k\pi}{n+1}$; $k = 0, 1, 2, \dots, n$

$n\varphi + 2k_1\pi = -\varphi + 2k_2\pi$; $k_1, k_2 \in \mathbb{Z}$

$n\varphi = -\varphi + 2(k_2 - k_1)\pi$

$n = 1$

$\varphi \in \{0, \pi\}$

Rešitev je realna os.

$n \geq 2$

Ena rešitev: $z = 0$

Druga rešitev: $|z| = 1$

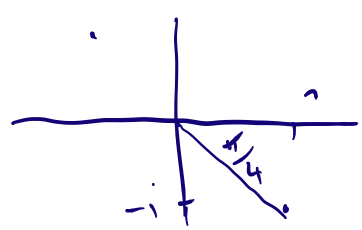
Za rešitev dobimo primitivne korene enote.

- $\underbrace{z^3 + 3z^2 + 3z} + i = 0$

$z^3 + 3z^2 + 3z + 1 - 1 =$

$= (z+1)^3$

$\underbrace{(z+1)}_{w}{}^3 = 1 - i$



$w = z + 1$

$w^3 = 1 - i$

$|w|^3 (\cos(3\varphi) + i\sin(3\varphi)) = \sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right)$

to $|w| = \sqrt[6]{2}$

$$\varphi = \frac{7\pi}{12} + \frac{2k\pi}{3}; \qquad k = 0, 1, 2$$

če ten rešitvam:

$$w_1 = \sqrt[6]{2}\left(\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right)$$

$$\dots$$

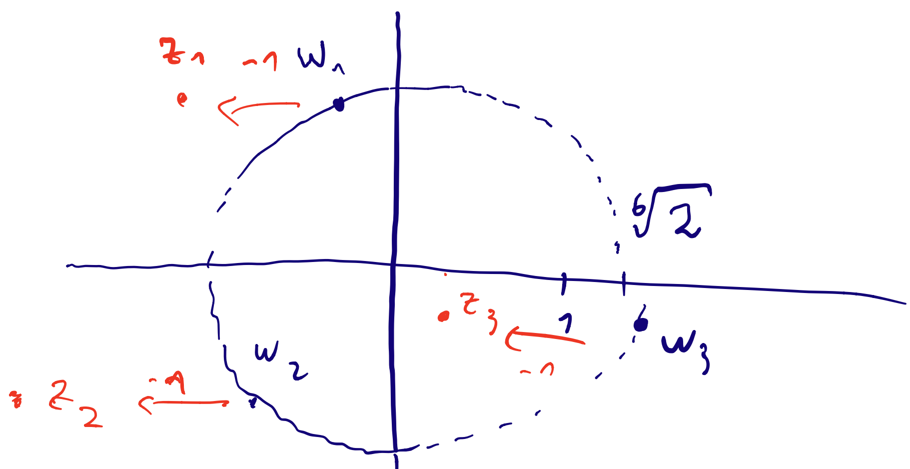
$$w_3 = \sqrt[6]{2}\left(\cos\left(\frac{7\pi}{12} + \frac{4\pi}{3}\right) + i\sin\left(\frac{7\pi}{12} + \frac{4\pi}{3}\right)\right)$$

$z_1 = w_1 - 1$

$z_2 = w_2 - 1$

$z_3 = w_3 - 1$

Narišimo rešitve v kompleksni ravnini!



$$\frac{7\pi}{12} + \frac{2\pi}{3} = \frac{15\pi}{12}$$

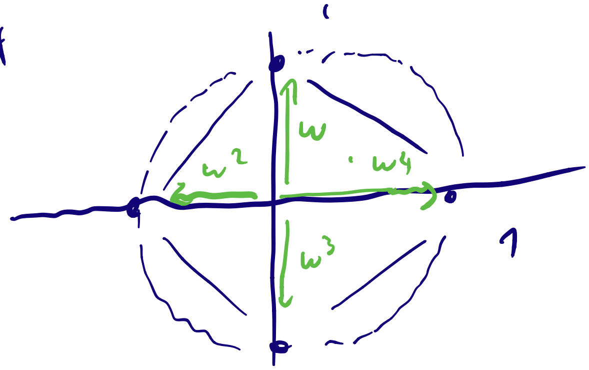
$$\frac{15\pi}{12} + \frac{2\pi}{3} = \frac{23\pi}{12}$$

Primitivni n-koreni enote ; $n \in \mathbb{N}$

To so $w \in \mathbb{C} : w^n = 1$

$$\underbrace{\cos(\varphi_1) + i\sin(\varphi_1)}_{e^{i\varphi_1}}, \ldots, \underbrace{\cos(\varphi_n) + i\sin(\varphi_n)}_{e^{i\varphi_n}}$$

$n = 4$



Rešitve so oglišča n-kotnika,

Vse rešitve so potence primitivnega korena enote:

$$\underset{w^n}{\underset{\parallel}{1}}, w, w^2, \ldots, w^{n-1}$$

$$1 + w + w^2 + w^3 + \dots + w^{n-1} = 0 \quad (n \neq 1)$$

Geometrijska rešitev: to je vsota vektorjev ~~od~~ oglišč n-kotnika.

Računsko:

$$1 + x + x^2 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

$x \neq 1$

$x \rightsquigarrow w$

$\underline{w \neq 1}$

$\Rightarrow 1 - w \neq 0$

$$1 + w + \dots + w^{n-1} = \frac{1 - w^n}{1 - w} = 0$$

$\underline{w = 1}$

$1 = 1$

Reši neenačbo z abs. vrednostmi v $\mathbb{R}$

$$|x + |2x+4|| \geq 2$$

Ločimo primere:

1. $2x + 4 \geq 0$

$$2x \geq -4$$

$$x \geq -2$$

$$|x + 2x + 4| \geq 2$$

$$|3x + 4| \geq 2$$

1.1. $3x + 4 \geq 0$

$$x \geq -\frac{4}{3}$$

$$3x + 4 \geq 2$$

$$3x \geq -2$$

$$x \geq -\frac{2}{3}$$

$\left[-\frac{2}{3}, \infty\right)$ je del rešitve

1.2 $3x + 4 < 0$

$$3x < -4$$

$$x < -\frac{4}{3}$$

$$-3x - 4 \geq 2$$
$$3x + 4 \leq -2$$
$$3x \leq -6$$
$$x \leq -2$$

$\{-2\}$ reši neenakost

2. $2x + 4 < 0$

$$x < -2 \quad \notin$$

$$|x - 2x - 4| \geq 2$$
$$|x+4| = |-x-4| \geq 2$$

2.1 $x + 4 > 0$

$$x > -4$$

zato, ker sem tu imela strogo neenakost

$$x + 4 \geq 2$$
$$x \geq -2$$

$2 \notin$

2.2 $x + 4 \leq 0$

$$x \leq -4$$
$$-x - 4 \geq 2$$
$$-x \geq 6$$
$$x \leq -6$$

$(-\infty, -6]$ je del rešitve

Množica rešitev je torej

$(-\infty, -6] \cup \{-2\} \cup [-\frac{2}{3}, \infty)$

Zaporedja realnih števil

$\{a_1, a_2, a_3, \dots \mid a_n \in \mathbb{R}\}$

$f: \mathbb{N}_+ \longrightarrow \mathbb{R}$

$f(n) := a_n$

(navzdol)

Zaporedje je navzgor omejeno, če

$\exists M \in \mathbb{R}$, da je $a_n \leq M \quad \forall n \in \mathbb{N}$

$(\geq)$

(strogo)

Zaporedje je naraščajoče, če je

$a_n \leq a_{n+1}$; $\forall n \in \mathbb{N}$

$(<)$

Dokaži, da je zaporedje $\{a_n\}_{n \geq 1}$ naraščajoče in navzgor neomejeno,

$$a_n = \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \dots + \frac{1}{\sqrt{2n}}$$

$(a_n)_n$ je navzgor neomejeno:

V a_n je $n+1$ členov, vsi so $\geq \frac{1}{\sqrt{2n}}$

$$a_n \geq \frac{n+1}{\sqrt{2n}} \geq \frac{n}{\sqrt{2n}} = \sqrt{\frac{n}{2}}$$

zaporedje $\left\{\sqrt{\frac{n}{2}}\right\}_n$ je navzgor neomejeno,

zato je tudi $(a_n)_n$ navzgor neomejeno.