LINEARNA ALGEBRA 2020/21
3. VAJE: 19. 10. 2020

4. ~~1.~~ Reši enačbo $\langle\vec{a},\vec{x}\rangle\vec{a}\times\vec{b} = \vec{a}\times\vec{x}$ za linerano neodvisna vektorja $\vec{a},\vec{b}$. Rešitev podaj parametrično.

3. ~~2.~~ Naj bo mešani produkt definiran z $[\vec{a},\vec{b}\times\vec{c}] = \langle\vec{a},\vec{b}\times\vec{c}\rangle$. Pokaži, da je $[\vec{a},\vec{b},\vec{c}] = [\vec{c},\vec{a},\vec{b}] = -[\vec{b},\vec{a},\vec{c}]$.

1 ~~3.~~ Izračunaj prostornino paralepipeda določenega z $\vec{a} = (1,2,0), \vec{b} = (0,3,-1)$ in $\vec{c} = (3,-2,4)$.

2 ~~4.~~ Pokaži, da so vektorji $\vec{a},\vec{b},\vec{c}$ linearno neodvisni natanko tedaj, ko je $[\vec{a},\vec{b},\vec{c}] \neq 0$.

5. Premicama $\vec{r} = (1,2,0) + t(1,1,1)$ in $\vec{s} = (2,3,1) + t(0,3,-1)$ določi normalno obliko. Ali se premici sekata?

6. Določi razdaljo od točke $A = (7,6,10)$ do premice, ki gre skozi $B = (3,7,0)$ in $C = (4,9,-2)$.

7. Določi parametrično in normalno enačbo ravnine, ki gre skozi točke $A = (-1,1,3), B = (1,4,-1)$ in $C = (2,-3,1)$.

8. Naj bo ravnina podana z normalo $\vec{n} = (7,1,0)$ in točko $x_0 = (2,3,-1)$. Zrcali točko $A = (4,3,5)$ čez ravnino in določi razdaljo od točke do zrcaljene točke.

$[a,b,c] = \langle a,\vec{b}\times\vec{c}\rangle = \pm$ "prostornina paralepipeda $\vec{a},\vec{b},\vec{c}$"

$$\begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{matrix} \qquad a_1(b_2c_3 - b_3c_2) + a_2(\quad)$$

prva komponenta $\vec{b}\times\vec{c}$

$$[(1,2,0),(0,3,-1),(3,-2,4)] = 4-6+0-0-2-0 = \underline{-4}$$

$$\begin{matrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 3 & -1 & 0 & 3 \\ 3 & -2 & 4 & 3 & -2 \end{matrix}$$

prostornina $= 4$

Pokaži, da so vektorji $\vec{a}, \vec{b}, \vec{c}$ linearno neodvisni natanko tedaj, ko je $[\vec{a}, \vec{b}, \vec{c}] \neq 0$.

V $\mathbb{R}^2$: $\vec{a}, b$ linearno odvisna $\iff a_1b_2 - a_2b_1 = 0$

V $\mathbb{R}^3$ $\vec{a}, \vec{b}$ lin odvisna $\iff \vec{a} \times \vec{b} = 0$, $|\vec{a} \times \vec{b}|$ – povrsina paralelograma

- $\vec{a}, \vec{b}, \vec{c}$ lin odvisni $\iff$ ležijo na isti ravnini

$$\Updownarrow$$

- $[\vec{a}, \vec{b}, \vec{c}] = 0 \iff$ vol paralepipeda $= 0 \iff \vec{a}, \vec{b}, \vec{c}$ so na isti ravnini

3. naloga

$$[\vec{a}, \vec{b}, \vec{c}] \overset{(2)}{=} -[\vec{b}, \vec{a}, \vec{c}] = -[\vec{a}, \vec{c}, \vec{b}] = [\vec{b}, \vec{c}, \vec{a}] \overset{(1)}{=} -[\vec{c}, b, \vec{a}] \overset{(2)}{=} [\vec{c}, \vec{a}, \vec{b}]$$

(1) $[\vec{a}, \vec{b}, \vec{c}] = -[\vec{a}, \vec{c}, \vec{b}]$ $\qquad \vec{c} \times \vec{b} = -\vec{b} \times \vec{c}$

$$\| \qquad\qquad \|$$

$$\langle \vec{a}, \vec{b} \times \vec{c} \rangle = -\langle a, \vec{c} \times \vec{b} \rangle$$

(2) $[\vec{a}, \vec{b}, \vec{c}] = -[\vec{b}, \vec{a}, \vec{c}]$

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array} \qquad \begin{array}{ccccc} b_1 & b_2 & b_3 & b_1 & b_2 \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}$$

$b_1 a_2 c_3 \to 213$

$\underline{a_i b_j c_k} \to ijk$

$+\ 123 \quad 231 \quad 312 \quad | \quad 213 \quad 321 \quad 132$

$-\ 321 \quad 132 \quad 213 \quad | \quad 231 \quad 312 \quad 123$

Kdaj je $(a \times b) \times c = a \times (b \times c)$?

$(\vec{a} \times \vec{b}) \times \vec{c} = \langle \vec{a}, \vec{c} \rangle \cdot \vec{b} - \langle \vec{b}, \vec{c} \rangle \cdot \vec{a}$ $\qquad a \times (\vec{b} \times \vec{c}) = -(\vec{b} \times \vec{c}) \times a = \langle \vec{a}, \vec{c} \rangle \vec{b} - \langle a, b \rangle \cdot \vec{c}$

enakost dokazano z računom s koordinatami

$\cancel{\langle \vec a, \vec c\rangle}\cdot \vec b - \langle \vec b, \vec c\rangle \cdot \vec a = \cancel{\langle \vec a, \vec c\rangle} \vec b - \langle a, \vec b\rangle \cdot \vec c$

$\langle \vec b, \vec c\rangle \vec a = \langle \vec a, \vec b\rangle \cdot \vec c$

možnosti:

1. $\langle \vec a, \vec b\rangle \neq 0 \to \vec c = \frac{\langle \vec b, \vec c\rangle}{\langle \vec a, \vec b\rangle}\vec a \iff \vec c = \lambda \vec a$ za nek λ
2. $\langle \vec b, \vec c\rangle \neq 0 \to \vec a = \frac{\langle \vec a, \vec b\rangle}{\langle \vec b, \vec c\rangle}\vec b \iff a = \lambda b$ za nek λ
3. $\langle \vec a, \vec b\rangle = \langle \vec b, \vec c\rangle = 0 \to b \perp \vec a, \vec c$

Reši enačbo $\langle \vec{a}, \vec{x}\rangle \vec{a} \times \vec{b} = \vec{a} \times \vec{x}$ za linerano neodvisna vektorja $\vec{a}, \vec{b}$. Rešitev podaj parametrično.

$\vec x$ zapišeno z neko bazo.

$\vec a, \vec b$ lin neodvisna $\quad \vec a \times \vec b \neq 0 \implies a, b, \vec a \times \vec b$ so baza

$\vec x = \lambda a + \mu \vec b + \gamma \vec a \times \vec b$

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skalarno množimo z x

$\langle a, \vec x\rangle \vec a \times \vec b = \vec a \times \vec x \quad \langle \cdot, x\rangle$

$\langle a, x\rangle [a, b, x] = 0 \quad \langle a \times b, x\rangle = [x, a, b] = [a, b, x]$

$\langle a, x\rangle = 0$ ali • $\boxed{[a, b, x] = 0}$

$\Downarrow$ iz desne strani enačbe

$\vec a \times \vec x = 0 \iff \vec a, \vec x$ lin odvisna $\quad$ en od njiju je 0

$\vec a, \vec x$ pravokotna $\implies x = 0$

Če je $\gamma = 0$, potem je $\lambda' \vec a + \mu' \vec b = 0$. Ampak $\vec a, b$ sta lin neodvisna. Torej je $\gamma \neq 0$

• $[a, b, x] = 0 \iff \vec a, \vec b, \vec x$ lin. odvisni $\quad \exists \lambda', \mu', \gamma'$ (ne vsi enaki 0) $\lambda' \vec a + \mu' \vec b + \gamma' x = 0 \implies \vec x = \lambda \vec a + \mu \vec b$

$\lambda = -\frac{\lambda'}{\gamma} \quad \mu = \frac{-\mu'}{\gamma}$

Vstavimo v enačbo.

$(\lambda \langle \vec a, \vec a\rangle + \mu \langle a, b\rangle)\, \vec a \times \vec b = \lambda\, \vec a \times \vec a + \mu\, \vec a \times \vec b \quad (\vec a \times \vec b \neq 0)$

$\langle a, \vec{\lambda a + \mu b}\rangle \vec a \times \vec b \qquad \overset{\shortparallel}{0} \qquad a \times (\lambda a + \mu b)$

$\lambda \langle \vec a, \vec a\rangle + \mu \langle \vec a, \vec b\rangle = \mu \to \lambda \langle \vec a, \vec a\rangle = (1 - \langle \vec a, \vec b\rangle)\mu$

$$\lambda = \frac{1 - \langle \vec a, \vec b\rangle}{\langle \vec a, \vec a\rangle}\mu$$

$\vec x = \frac{1 - \langle a, b\rangle}{\langle a, a\rangle}\mu\, \vec a + \mu \vec b$ – 1-parametrična družina rešitev

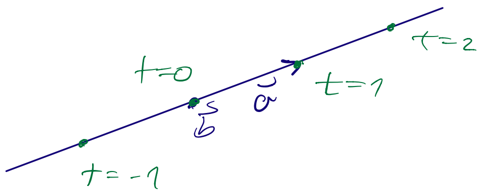
5. Premicama $\vec{r} = (1,2,0) + t(1,1,1)$ in $\vec{s} = (2,3,1) + t(0,3,-1)$ določi normalno obliko. Ali se premici sekata?

$\mathbb{R}^n$ $\quad \vec{b}$ - točka na premici $\quad$ parametrična $\vec{a}\cdot t + \vec{b}$ $\quad \vec{a} = (a_1, \dots, a_n)$

$\vec{a}$ - smerni vektor

$a_1 \cdot t + b_1, \; a_2 \cdot t + b_2, \; \dots$

$a_1 \cdot t + b_1 = x_1$



$$\frac{x_1 - b_1}{a_1} = \frac{x_2 - b_2}{a_2} = \dots \frac{x_n - b_n}{a_n} = t$$

$a_i = 0 \rightsquigarrow x_i = b_i$ (nadomestimo)

normalna oblika

$\vec{r} = (1,1,1)\cdot t + (1,2,0)$ $\qquad \vec{s} = (0,3,-1)t + (2,3,1)$

$$\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-0}{1} \qquad x = 2 \quad \frac{y-3}{3} = \frac{z-1}{-1}$$

imenovalci $\qquad$ - števci brez (x,y,z)

$(1,1,1)\cdot t + (1,2,0)$

$$(1,1,1)\cdot t_1 + (1,2,0) = (0,3,-1)\,t_2 + (2,3,1)$$

- $1t_1 + 1 = 0t_2 + 2$ $\qquad$ $t_1 = 1$
- $t_1 + 2 = 3t_2 + 3$ $\qquad$ $3 = 3t_2 + 3 \rightsquigarrow t_2 = 0$
- $1t_1 = -t_2 + 1$ $\qquad$ $1 = 1$ ✓

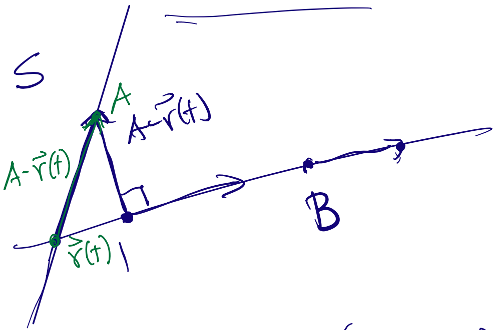
Presečišče $\quad \vec{r}(1) = \vec{s}(0) = (1,2,0) + (1,1,1) = (2,3,1)$.

- Splošno: Če je c_1, c_2 rešitev sistema enačb $\vec{r}(t_1) = \vec{s}(t_2)$, potem je $r(c_1)$ presečišče. Če sistem ni rešljiv, presečišča ni.

tč.

eka

6. Določi razdaljo od točke $A = (7, 6, 10)$ do premice, ki gre skozi $B = (3, 7, 0)$ in $C = (4, 9, -2)$.



$$r = (C-B)\cdot t + B$$

$$\vec{r} = (1, 2, -2)\cdot t + (3, 7, 0)$$

$$s(t) = A - \vec{r}(t) = (-1, -2, 2)\cdot t + (4, -1, 10)$$

Iščemo točko, kjer je

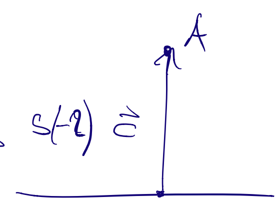
$$s(t) \perp \vec{r}(t) \qquad \langle A - r(t), (1, 2, -2)\rangle = 0$$

$$\langle (-1, -2, 2)t + (4, -1, 10), (1, 2, -2)\rangle = 0$$

$$-t\langle (1, 2, -2), (1, 2, -2)\rangle + \langle (4, -1, 10), (1, 2, -2)\rangle = 0$$

$$-t(1 + 4 + 4) + (4 - 2 - 20) = 0$$

$$-9t - 18 = 0 \qquad 9t = -18 \qquad \boxed{t = -2}$$



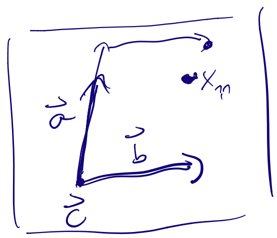
$$S(-2) = (2, 4, -4) + (4, -1, 10) = (6, 3, 6) = \vec{c}$$

$$\|\vec{c}\| = \text{"razdalja od točke do premice"} = \sqrt{2\cdot 36 + 9} = \sqrt{72 + 9} = \sqrt{81} = \boxed{9}$$

7. Določi parametrično in normalno enačbo ravnine, ki gre skozi točke $A = (-1, 1, 3)$, $B = (1, 4, -1)$ in $C = (2, -3, 1)$.

V $\mathbb{R}^3$

parametrična oblika



poljubna točka na premici

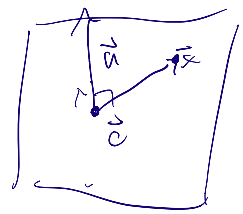
$$t_1\vec a + t_2\vec b + \vec c = \vec x$$

točka na ravnini

t_1, t_2 parametra

$$1\vec a + 1\vec b + \vec c = x_n$$

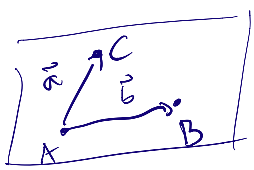
normalna oblika



$\vec n \perp \vec x - \vec c$

$\Updownarrow$

$\vec x$ je na ravnini



$\vec a = ?$ $\quad C - A = (3, -4, -2)$

$\vec b = ?$ $\quad B - A = (2, 3, -4)$

$\vec c = ?$ $\quad A, B, C = (2, -3, 1)$

$\vec n$ − normala

$$\langle \vec n, \vec x - \vec c\rangle = 0$$

normalna enačba ravnine

$$t_1(3, -4, -2) + t_2(2, 3, -4) + (2, -3, 1)$$ parametrična enačba

normalna enačba

$\vec a \times \vec b \perp \vec a, \vec b$

$$\vec n = (3, -4, -2) \times (2, 3, -4) = (16 + 6, 12 - 4, 9 + 8) = (22, 8, 17)$$

$$\langle \vec n, \vec x - (2, -3, 1)\rangle = 0 \qquad \vec x = (x, y, z)$$

$$22x + 8y + 17z = \langle (22, 8, 17), (2, -3, 1)\rangle$$

Ravnina je množica rešitev ene linearne enačbe.