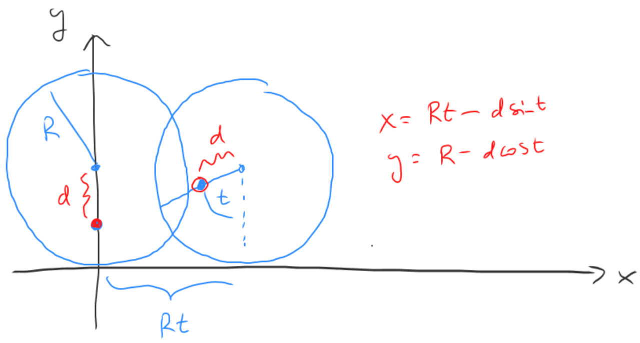
1

Domače naloge

NALOGA 1.1. Parametriziraj pot, ki jo opiše točka na razdalji d od središča kroga radija R, ko se le ta brez zdrsavanja kotali po premici.

REŠITEV.

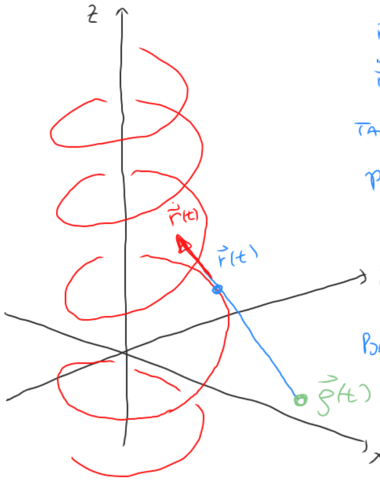


NALOGA 1.2. Dana naj bo spirala S $(a, b > 0)$ s parametrizacijo

$$\vec{r}(t) = (a \cos t, a \sin t, bt) \qquad t \in \mathbb{R}\,.$$

Parametriziraj krivuljo, ki jo opiše presečišče tangentne premice na S skozi točko $\vec{r}(t)$ z xy-ravnino.

REŠITEV.



$\vec r = (a\cos t, a\sin t, bt)$

$\dot{\vec r} = (-a\sin t, a\cos t, b)$

TANG. PREMICA

$p: \ \vec r(t) + s\,\dot{\vec r}(t)$

$\Rightarrow \ x = a\cos t - as\sin t$

$y = a\sin t + as\cos t$

$z = bt + sb$

$P_{(xy)}$ $\boxed{z=0} \Rightarrow s = -t$ in dolžino

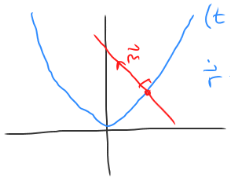
$$\boxed{\vec\rho(t) = (a\cos t + at\sin t,\ a\sin t - at\cos t,\ 0)}$$

Naloga 1.3. Dana je krivulja K s parametrizacijo

$$\vec r(t) = (t, \cosh t - 1) \qquad t \in \mathbb{R}\,.$$

- Izračunaj dolžino dela krivulje K za $t \in [0,1]$.
- Krivuljo K naravno parametriziraj.
- Naj bo $y_0(t)$ točka na y-osi, kjer normala na krivuljo K skozi točko $\vec r(t)$ seka y-os. Izračunaj
$$\lim_{t\to 0} y_0(t)\,.$$

Rešitev.



$(t, \operatorname{ch} t - 1)$

$\dot{\vec r} = (1, \operatorname{sh} t)$

$|\dot{\vec r}| = \operatorname{ch} t$

a) $\int_0^1 \operatorname{ch} t\, dt = \operatorname{sh} 1$ ✓

b) $s = \int_0^t \operatorname{ch} \tau\, d\tau = \operatorname{sh} t$

$t = \operatorname{arsh} s$

$\vec\rho(s) = (\operatorname{arsh} s, \operatorname{ch}(\operatorname{arsh} s) - 1)$

NAR. PAR.

c) $\vec n = (-\operatorname{sh} t, 1)$

NORM. PREMICA

$(t, \operatorname{ch} t - 1) + s(-\operatorname{sh} t, 1)$

$x = t - s\operatorname{sh} t \quad \leadsto x = 0 \Rightarrow s = \frac{t}{\operatorname{sh} t}$

$y = \operatorname{ch} t - 1 + s$

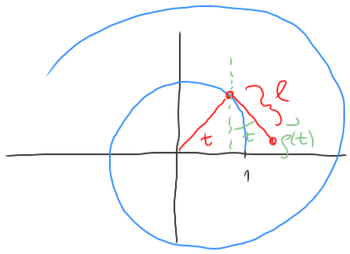
$y = \operatorname{ch} t - 1 + \frac{t}{\operatorname{sh} t} \xrightarrow[y\to 0]{\lim} \boxed{1}$

Naloga 1.4. Dana je spirala v ravnini s parametrizacijo

$$\vec r(t) = (e^t \cos t, e^t \sin t) \qquad t \in [0, \infty)\,.$$

Parametriziraj pot, ki jo opiše konec niti z začetkom v točki $(1, 0)$, ko se le ta (t.j. nit) odvija z dane spirale.

REŠITEV.



$\vec r = (e^t\cos t, e^t\sin t)$ $\quad t\in[0,\infty)$

$\dot{\vec r} = (e^t\cos t - e^t\sin t, e^t\cos t + e^t\sin t)$

$|\dot{\vec r}| = \sqrt{e^{2t} + e^{2t}} = \sqrt{2}\,e^t$

$\ell = \int_0^t \sqrt{2}\,e^\tau d\tau = \sqrt{2}(e^t - 1)$

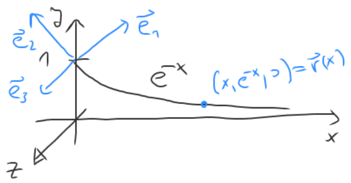
$$\vec\rho(t) = \left(e^t\cos t + \sqrt{2}(e^t-1)\sin t,\ e^t\sin t - \sqrt{2}(e^t-1)\cos t\right)$$
$$t\in[0,\infty)$$

NALOGA 1.5. Dana naj bo funkcija

$$f : [0, \infty) \to \mathbb{R} \qquad f(x) = e^{-x}\,.$$

- Parametriziraj ploskev, ki jo dobimo, če graf funkcije f zavrtimo okoli x-osi.
- Parametriziraj ploskev, ki jo dobimo, če graf funkcije f zavrtimo okoli y-osi.
- Parametriziraj ploskev, ki jo dobimo, če graf funkcije f zavrtimo okoli premice $y = x + 1$.

REŠITEV.



a) $\vec r(x,\varphi) = (x, e^{-x}\cos\varphi, e^{-x}\sin\varphi)$

b) $\vec r(x,\varphi) = (e^{-x}\cos\varphi, e^{-x}\sin\varphi, e^{-x})$

c) $\vec e_1 = (\frac{1}{\sqrt2}, \frac{1}{\sqrt2}, 0)$ $\qquad \vec\rho(x,\varphi) = (\vec r\cdot\vec e_1)\vec e_1 + (\vec r\cdot\vec e_2)\cos\varphi\,\vec e_2 +$

$\vec e_2 = (-\frac{1}{\sqrt2}, \frac{1}{\sqrt2}, 0)$ $\qquad + (\vec r\cdot\vec e_2)\sin\varphi\,\vec e_3\,.$

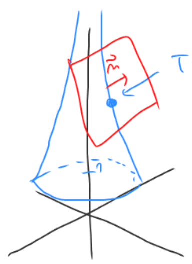
$\vec e_3 = (0, 0, 1)$

NALOGA 1.6. Dana je ploskev S s parametrizacijo

$$\vec{r}(\varphi, x) = (x\cos\varphi, x\sin\varphi, \tfrac{1}{x}) \qquad \varphi \in [0, 2\pi],\ x \in (0, 1]\,.$$

Določi enačbo tangentne ravnine na ploskev S v točki $\vec{r}(\frac{\pi}{4}, \frac{1}{2})$.

REŠITEV.



$T = \vec{r}(\pi/4, \tfrac{1}{2}) = \left(\tfrac{\sqrt{2}}{4}, \tfrac{\sqrt{2}}{4}, 2\right)$

$\vec{r}_\varphi = (-x\sin\varphi, x\cos\varphi, 0) = \left(-\tfrac{\sqrt{2}}{4}, \tfrac{\sqrt{2}}{4}, 0\right)$ × VEKT. PROD

$\vec{r}_x = (\cos\varphi, \sin\varphi, -\tfrac{1}{x^2}) = \left(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, -4\right)$

$\vec{n} = \left(-\sqrt{2}, -\sqrt{2}, -\tfrac{1}{2}\right)$

$$\boxed{\sqrt{2}x + \sqrt{2}y + \tfrac{1}{2}z = 2}$$

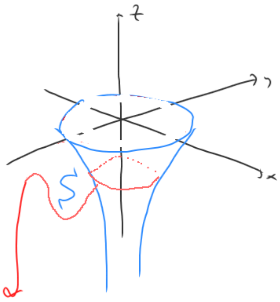
TANG. RAVNINA

NALOGA 1.7. Dana je ploskev S s parametrizacijo

$$\vec{r}(\varphi, x) = (x\cos\varphi, x\sin\varphi, \ln x) \qquad \varphi \in [0, 2\pi],\ x \in (0, 1]\,.$$

Določi vse točke na ploskvi S, katerih tangentna ravnina seka xy-ravnino pod kotom $\frac{\pi}{3}$.

REŠITEV.



VSE TE TOČKE SO REŠ.

$\vec{r}_\varphi = (-x\sin\varphi, x\cos\varphi, 0)$

$\vec{r}_x = (\cos\varphi, \sin\varphi, \tfrac{1}{x})$

$\Downarrow$ VEKT. PROD. ×

$\vec{n} = (\cos\varphi, \sin\varphi, -x)$

$\Rightarrow$ ENOTSKA NORMALA $\vec{n} = \frac{1}{\sqrt{1+x^2}}(\cos\varphi, \sin\varphi, +x)$

Pogoj

$\vec{n}\cdot\vec{k} = \cos\frac{\pi}{3} = \frac{1}{2}$

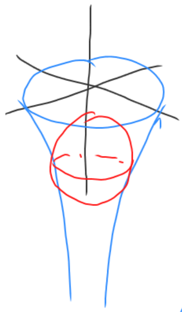
$+\frac{x}{\sqrt{1+x^2}} = \frac{1}{2} \Longrightarrow x = \frac{1}{\sqrt{3}}$ $\quad\varphi$ polj.

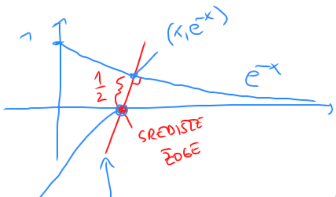
NALOGA 1.8. Dana je ploskev S v obliki lijaka s parametrizacijo

$$\vec{r}(\varphi, z) = (e^z\cos\varphi, e^z\sin\varphi, z) \qquad \varphi \in [0, 2\pi],\ z \in (-\infty, 0]\,.$$

V lijak vržemo žogo premera 1. Določi z-koordinato središča žoge, ko se ta ustavi v lijaku.

Rešitev.





$(x, e^{-x}) + t(e^{-x}, 1) =$ /

$e^{-x} + t = 0 \Rightarrow t = -e^{-x}$

$(x - e^{-2x}, 0)$

$|(e^{-2x}, +e^{-x})| = \frac{1}{2}$

$e^{-4x} + e^{-2x} = \frac{1}{4} \Rightarrow e^{-2x} = \frac{\sqrt{2}-1}{2}$

$\Downarrow$

$x = -\frac{1}{2}\ln\frac{\sqrt{2}-1}{2} \Rightarrow (x - e^{-2x}) = -\frac{1}{2}\ln\frac{\sqrt{2}-1}{2} - e^{\ln\frac{\sqrt{2}-1}{2}}$ $\boxtimes$

Naloga 1.9. Naj bo $I(x) = \int_x^{x^2} e^{-xt^2}\,dt$. Izračunaj $I'(x)$!

Rešitev.

$$I = \int_x^{x^2} e^{-xt^2}\,dt$$

$$I' = e^{-x^5} - e^{-x^3} + \int_x^{x^2} -t^2 e^{-xt^2}\,dt = \dots$$

Naloga 1.10. Ali konvergirata enakomerno dana integrala:

$$\int_0^\infty \frac{e^{-mx}}{1+x^2} \qquad \text{za } m \geq 0$$

$$\int_{-\infty}^\infty \frac{\cos mx}{1+x^2} \qquad \text{za } m \in (-\infty, \infty)$$

Rešitev.

$$\int_0^\infty \frac{e^{-mx}}{1+x^2}\,dx \qquad \left|\frac{e^{-mx}}{1+x^2}\right| \leq \frac{1}{1+x^2} \qquad m \in [0,\infty)$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx \quad \text{KONV.}$$

$\Downarrow$

KONV. ENAK. na $[0,\infty)$

$$\int_0^\infty \frac{\cos mx}{1+x^2}\,dx \qquad \left|\frac{\cos mx}{1+x^2}\right| \leq \frac{1}{1+x^2} \qquad m \in \mathbb{R}$$

$$\int_0^\infty \frac{dx}{1+x^2} \quad \text{KONV.}$$

$\Downarrow$

KONV. ENAK. na $\mathbb{R}$

NALOGA 1.11. Ali konvergira enakomerno za $y \in [1, \infty]$ integral

$$F(y) = \int_0^\infty ye^{-xy}dx\,?$$

Kaj pa za $y \in (0, 1)$?

REŠITEV.

$$F(y) = \int_0^\infty y e^{-xy}\,dx$$

Naj bo $y \in [1,\infty)$ in $\varepsilon > 0$ polj.

$$\left|\int_M^\infty y e^{-xy}\,dx\right| = e^{-My} \leq e^{-M} < \varepsilon \qquad y \in [1,\infty)$$

za $M > -\ln \varepsilon$

$\Rightarrow$ KONV. ENAK. NA $[1,\infty)$

Naj bo $y \in (0,1)$

M polj. velik

$$\left|\int_M^\infty y e^{-xy}\,dx\right| = e^{-My} \xrightarrow[y\to 0]{\lim} 1 \Rightarrow \text{NI MAJHNO za vse } y \in (0,1)$$

(NEGLEDE NA TO KAKO VELIK JE M)

NALOGA 1.12. Dan je integral s parametrom

$$J(y) = \int_1^\infty \frac{y^x}{xy^x + x^2}\,dx\,.$$

Za katere $y \geq 0$ integral konvergira in pokaži, da konvergira enakomerno na intervalu $[0, 1]$. Ali lahko izračunamo $J'(a)$ s pomočjo metode odvajanja pod integralom?

Rešitev.

$J(y) = \int_1^\infty \frac{y^x}{xy^x + x^2}\,dx$ za katere $y \geq 0$ konv.?

$y=0$: $\int_1^\infty \frac{0}{0+x^2}\,dx = 0 \Rightarrow J(0) = 0$

$y=1$: $\int_1^\infty \frac{1}{x+x^2}\,dx \leq \int_1^\infty \frac{dx}{x^2}$ konv ✓

$y \in (0,1)$ $\lim_{x\to\infty} \frac{y^x}{x y^x + x^2} = 0$ $\rightsquigarrow$ $\int_1^\infty \frac{y^x}{xy^x+x^2}\,dx = \int_1^\infty \frac{\left(\frac{x^2 y^x}{xy^x+x^2}\right)}{x^2}\,dx$ KONV.

$\lim_{x\to\infty} \underbrace{\frac{x^2}{xy^x+x^2}}_{1} \cdot \underbrace{y^x}_{0} = 0$

$y > 1$ $\lim_{x\to\infty} \frac{y^x}{xy^x+x^2} = \frac{1}{x + \left(\frac{x^2}{y^x}\right)} = 0$

$\int_1^\infty \frac{y^x}{xy^x+x^2}\,dx = \int_1^\infty \frac{\frac{xy^x}{xy^x+x^2}}{x}\,dx$ $\lim_{x\to\infty} \frac{xy^x}{xy^x+x^2} = \lim_{x\to\infty} \frac{1}{1+\left(\frac{x}{y^x}\right)} = 1$

DIV.

Integral konv. $\Leftrightarrow$ $y \in [0,1]$

Velja $\left| \frac{y^x}{xy^x+x^2} \right| \leq \frac{1}{x^2}$ $\forall y \in [0,1]$ & $\int_1^\infty \frac{dx}{x^2}$ konv.

$\Rightarrow$ $J(y)$ ENAK. KONV. na $[0,1]$

$$J'(y) \overset{?}{=} \int_1^\infty \frac{xy^{x-1}(xy^x+x^2) - y^x \cdot x^2 y^{x-1}}{(xy^x+x^2)^2}\,dx =$$

$$= \int_1^\infty \frac{x^3 y^{x-1}}{(xy^x+x^2)^2}\,dx = \int_1^\infty \frac{xy^{x-1}}{(x+y^x)^2}\,dx$$

$y=1 \Rightarrow \int_1^\infty \frac{x\,dx}{(1+x)^2}$ DIVERGI

Če pa $y \in [0,1)$

imamo $\displaystyle\int_1^\infty \frac{x y^{x-1}}{(x+y^x)^2}\,dx = \int_1^\infty \frac{\frac{x^3 y^{x-1}}{(x+y^x)^2}}{x^2}\,dx$ KONVERGIRA, $\frac{x^3 y^{x-1}}{(x+y^x)^2} \xrightarrow{\lim_{x\to\infty}} 0$

Vzemimo

$y \in [0,c] \subset [0,1)$

$$\left| \frac{x y^{x-1}}{(x+y^x)^2} \right| \leq \frac{x c^{x-1}}{x^2} = \frac{c^{x-1}}{x} \leq c^{x-1} \quad \forall y \in [0,c]$$

$$\int_1^\infty c^{x-1}\,dx = \frac{c^{x-1}}{\ln c}\Big|_1^\infty = \frac{1}{\ln c} \quad \text{KONVERGIRA}$$

$\Rightarrow \displaystyle\int_1^\infty \frac{x y^{x-1}}{(x+y^x)^2}\,dx$ konv. enak. na $[0,c]$ $\Rightarrow$

$$J'(y) = \int_1^\infty \frac{x y^{x-1}}{(x+y^x)^2}\,dx \quad \text{na } y \in [0,c] \subset [0,1) \quad (\text{za polj. } c)$$

torej $\displaystyle J'(y) = \int_1^\infty \frac{x y^{x-1}}{(x+y^x)^2}\,dx$ za $\forall\, y \in [0,1)$.

Naloga 1.13. Dan je integral s parametrom

$$J(y) = \int_1^\infty \frac{\log x (2y)^x}{x}\,dx\,.$$

Za katere $y \geq 0$ integral konvergira in pokaži, da konvergira enakomerno na vsakem intervalu $[0,c] \subset [0,\frac{1}{2})$. Ali lahko izračunamo $J'(a)$ s pomočjo metode odvajanja pod integralom?

Rešitev.

$$J(y)=\int_1^\infty \frac{\ln x \quad (2y)^x}{x}\,dx$$

$y=0$: $J(0)=0$ konv. za $y=0$

$y=\frac{1}{2}$: $\int_1^\infty \frac{\ln x}{x}\,dx$ DIV.

$y\in[0,\frac{1}{2})$: $\int_1^\infty \frac{\ln x\ (2y)^x}{x}\,dx=\int_1^\infty \frac{x\ln x\,(2y)^x}{x^2}\,dx$ KONV.

$0\le x\ln x\,(2y)^x\le x^2(2y)^x$

$\downarrow_{x\to\infty}$ $\Leftarrow$ $x\to\infty\downarrow$ $y\in[0,\frac{1}{2})$

0 $\quad$ 0

$y\geq\frac{1}{2}$: $\int_1^\infty \frac{\ln x\,(2y)^x}{x}\,dx$ DIV. $\quad \xrightarrow{x\to\infty}\infty$

Integral konvergira za $y\in[0,\frac{1}{2})$.

Naj bo $[0,c]\subset[0,\frac{1}{2})$

$$\left|\frac{\ln x\,(2y)^x}{x}\right|\le\frac{\ln x\ (2c)^x}{x}\quad \&\quad \int_1^\infty\frac{\ln x\,(2c)^x}{x}\,dx \text{ KONV.}$$

$\Rightarrow$ $J(y)$ konv. enak. na vsakem intervalu $[0,c]\subseteq[0,\frac{1}{2})$.

$$J'(y)\overset{?}{=}\int_1^\infty\frac{\ln x\cdot x\,(2y)^{x-1}}{x}\,dx=\int_1^\infty(2y)^{x-1}\ln x\,dx$$

$y\in[0,c]\subset[0,\frac{1}{2})\ \Rightarrow\ |(2y)^{x-1}\ln x|\le(2c)^{x-1}\ln x\quad\forall y\in[0,c]$

$$\int_1^\infty (2c)^{x-1}\ln x\,dx = \int_1^\infty \frac{x^2(2c)^{x-1}\ln x}{x^2}\,dx \quad \text{konv. saj}$$

$$0 \le \lim_{x\to\infty} x^2(2c)^{x-1}\ln x \le \underbrace{x^3(2c)^{x-1}}_{\to 0,\ x\to\infty}$$

$$\Rightarrow \quad J'(y) = \int_1^\infty (2y)^{x-1}\ln x\,dx \quad \text{na } [0, \tfrac{1}{2}).$$

NALOGA 1.14. Izračunaj $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx$.

REŠITEV.

$$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$$

Uvedemo $$I(a) = \int_0^1 \frac{\ln(1+ax)}{1+x^2}\,dx \qquad a\in[0,1]$$

$\rightsquigarrow$ DI posplošen

$$I'(a) = \int_0^1 \frac{x}{(1+ax)(1+x^2)}\,dx = \dots = \frac{a\pi+\ln 4}{4(1+a^2)} - \frac{\ln(1+a)}{1+a^2}$$

$I(0)=0$

$$\Rightarrow \quad I(1) = \int_0^1 \frac{a\pi+\ln 4}{4(1+a^2)} - \frac{\ln(1+a)}{1+a^2}\,da = \text{ZVIT TRIK} =$$

$$= \int_0^1 \frac{a\pi + \ln 4}{4(1+a^2)}\, da - \underbrace{\int_0^1 \frac{\ln(1+a)}{1+a^2}\, da}_{I(1)} =$$

$$= \int_0^1 \frac{a\pi + \ln 4}{4(1+a^2)}\, da - I(1) \Rightarrow$$

$$I(1) = \frac{1}{2} \int_0^1 \frac{a\pi + \ln 4}{4(1+a^2)}\, da = \ldots = \frac{\pi}{8} \ln 2$$

Naloga 1.15. Dana naj bo funkcija

$$I(m) = \int_0^1 \frac{m f(x)}{m^2 + x^2}\, dx\,,$$

kjer je f zvezna na $[0,1]$. Ali je I zvezna na $\mathbb{R}$?

Rešitev.

$$I(m) = \int_0^1 \frac{m f(x)}{m^2 + x^2}\, dx \quad , \quad f : [0,1] \to \mathbb{R} \text{ zvezna}$$

Naj bo $[c,d] \subset (0,\infty)$ ali $[c,d] \subset (-\infty, 0)$

Potem je $\frac{m f(x)}{m^2+x^2}$ zvezna na $\underset{x}{[0,1]} \times \underset{m}{[c,d]} \Rightarrow I$ zvezna na $[c,d]$

Ker sta c,d poljubna $\Rightarrow I$ zvezna na $\mathbb{R} - \{0\}$.

Težave so prisotne le pri $m = 0$.

Naj bo $m \in [-a, a]$

$$\left| \int_0^b \frac{m f(x)}{m^2+x^2} dx \right| \leq \max_{x\in[0,b]} |f| \cdot \int_0^b \frac{|m| dx}{m^2+x^2} =$$

$$= \max_{x\in[0,b]} |f| \operatorname{arctg} \frac{b}{|m|} \leq \max_{x\in[0,b]} |f| \cdot \frac{\pi}{2}$$

1) Če je $f(0)=0 \Rightarrow \exists b$ dovolj blizu 0, da je $\max_{x\in[0,b]} |f| \cdot \pi/2 < \epsilon$ za vsak $m \in [-a,a]$

$\Rightarrow$ I zvezna tudi v 0.

2) Če je $f(0) \neq 0$, potem I ni več zvezna v točki $m=0$.

Naj bo torej $f(0)\neq 0$. Velja $I(0)=0$, ker pa je

$$\lim_{m\to 0} I(m) = \lim_{m\to 0} \int_0^1 \frac{m f(x)}{m^2+x^2} dx \overset{\delta \text{ fiksen (majhen)}}{=}$$

$$= \lim_{m\to 0} \left(\int_0^\delta \frac{m f(x)}{m^2+x^2} dx + \underbrace{\int_\delta^1 \frac{m f(x)}{m^2+x^2} dx}_{\downarrow m\to 0 \atop 0} \right) =$$

Oglejmo si

$$\int_0^\delta \frac{m f(x)}{m^2+x^2}\,dx \,. \quad \text{Denimo } f(0)=2a>0 \Rightarrow$$

$\Rightarrow$ Za δ dovolj majhen je

$$\int_0^\delta \frac{m f(x)}{m^2+x^2}\,dx \geq a\int_0^\delta \frac{m\,dx}{m^2+x^2} = a \operatorname{arctg}\frac{x}{m}\Big|_0^\delta = a\operatorname{arctg}\frac{\delta}{m}$$

velja

$$\lim_{m\to 0} a \operatorname{arctg}\frac{\delta}{m} = \pm a\frac{\pi}{2} \neq 0 \quad ✓$$

Torej I ni zvezna v 0.

Naloga 1.16. Za katere $p \in \mathbb{R}$ konvergira integral

$$I(p) = \int_0^{\frac{\pi}{2}} \tan^p x \, dx \,.$$

Izračunaj I.

Rešitev.

$$I(p) = \int_0^{\pi/2} tg^p x\,dx = \int_0^{\pi/2}\frac{\sin^p x}{\cos^p x}\,dx \Rightarrow$$

$\Rightarrow$ Pri $x=0$ konv. $\Leftrightarrow$ $-p<1$ $\Rightarrow$ $\boxed{p>-1}$

Pri $x=\pi/2$ konv. $\Leftrightarrow$ $p<1$ $\Rightarrow$

Torej $\boxed{p\in(-1,1)}$ $\rightsquigarrow$ Tu integral konvergira.

$2-1-p$

$1-\frac{1+p}{2}$

$$I(p)=\int_0^{\pi/2}\sin^p x\cos^{-p}x = \frac{1}{2} B\left(\frac{1+p}{2},\frac{1-p}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(\frac{1-p}{2}\right)$$

$\frac{1+p}{2}\in(0,1)$

$$=\frac{1}{2}\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(1-\frac{1+p}{2}\right)$$

$$=\frac{1}{2}\frac{\pi}{\sin\left(\pi\left(\frac{1+p}{2}\right)\right)}$$

Naloga 1.17. Za katere $p, q \in \mathbb{R}$ konvergira integral

$$I(p,q) = \int_1^\infty (x^r - 1)^p x^{-q}\, dx \qquad r > 0\,.$$

Izrazi I s pomočjo Beta funkcije.

Rešitev.

$$I(p,q) = \int_1^\infty (x^r - 1)^p\, x^{-q}\, dx \quad , \; r>0$$

$t = x^r - 1$ $\qquad = \int_0^\infty t^p\, (1+t)^{-\frac{q}{r}} \cdot \frac{1}{r} (1+t)^{\frac{1}{r}-1} dt =$

$x^r = 1+t$

$x = (1+t)^{1/r}$ $\qquad = \frac{1}{r} \int_0^\infty t^p\, (1+t)^{\frac{1-r-q}{r}}\, dt =$

$dx = \frac{1}{r} (1+t)^{\frac{1}{r}-1}$

$$= \frac{1}{r} \int_0^\infty \frac{t^p}{(1+t)^{\frac{r+q-1}{r}}}\, dt =$$

$$= \frac{1}{r} B(a,b) \qquad a-1 = p \quad a+b = \frac{r+q-1}{r}$$

Naloga 1.18. Izračunaj dvojni integral

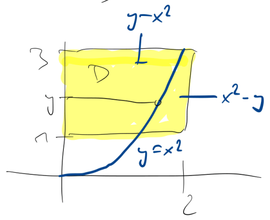
$$\iint\limits_D |x^2 - y|\, dx\, dy\,,$$

kjer je

$$D = [0,2] \times [1,3] \subset \mathbb{R}^2\,.$$

Rešitev.

$$\iint_D |x^2 - y| \, dx\, dy = \int_1^3 dy \int_0^{\sqrt{y}} y - x^2 \, dx \; +$$



$$+ \int_1^3 dy \int_{\sqrt{y}}^2 x^2 - y \, dx =$$

$$= \ldots$$