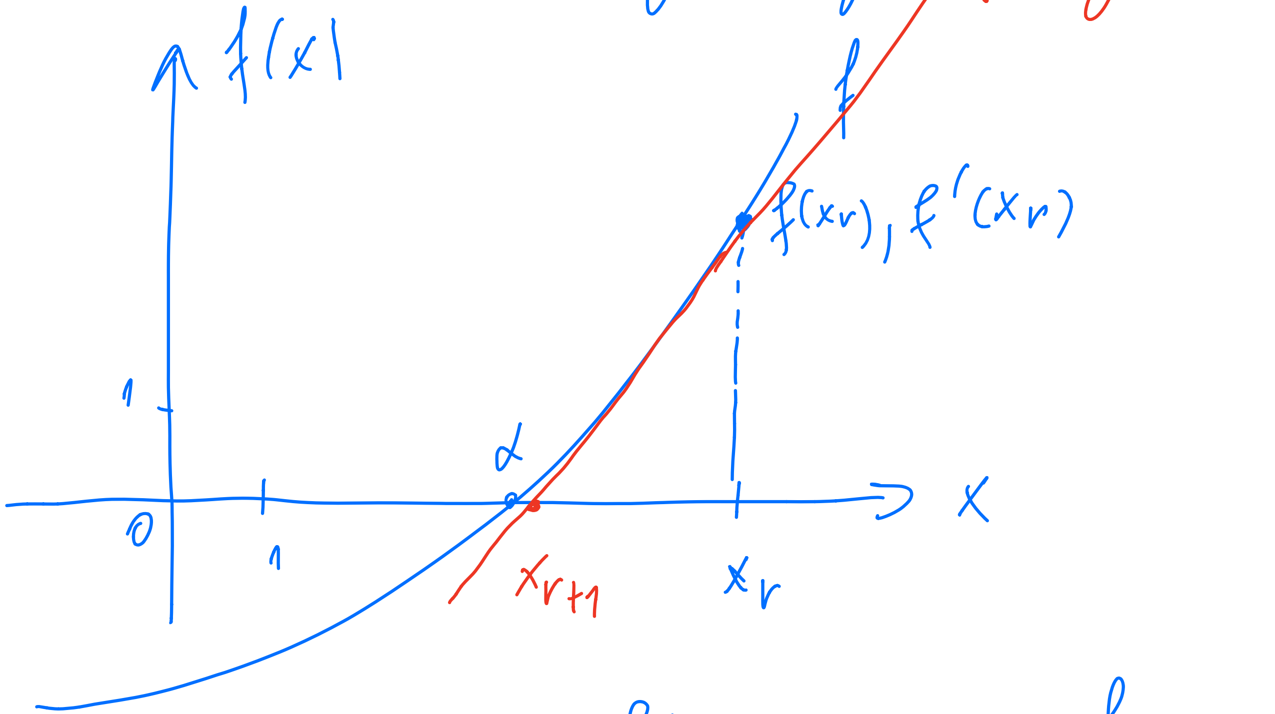
Reševanje nelinearnih enačb (nadaljevanje)



$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} = x_r - \frac{f_r}{f'_r}$$

Alternativna izpeljava:

Ideja: $x_r \to x_{r+1}$?

$$x_{r+1} = x_r + h \quad ; \quad |h| \ll 1$$

$$\underset{\text{želja}}{0} = f(x_{r+1}) = f(x_r + h) = f(x_r) + h f'(x_r) \underbrace{+ \frac{h^2}{2} f''(x_r) + \dots}_{\text{odrežemo}}$$

$$f(x_r) + h f'(x_r) \approx 0$$

$$h = -\frac{f(x_r)}{f'(x_r)}$$

$$f'(x_r)$$

$$x_{r+1} - x_r = -\frac{f(x_r)}{f'(x_r)}$$

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} \; ; \; r = 0, 1, \dots$$

<u>Izrek</u>: Naj bo α ničla gladke funkcije f v okolici točke α. Tangentna metoda

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} \; ; \; r = 0, 1, \dots$$

konvergira proti α z redom konvergence vsaj 2, če je α enostavna ničla f in x_0 izberemo blizu α.

<u>Dokaz</u>: Če je α enostavna ničla, je

$$f(x) = (x - \alpha)\, h(x) \; ; \; h(\alpha) \neq 0 .$$

Iteracijska funkcija tangentne metode je

$$g(x) = x - \frac{f(x)}{f'(x)} .$$

Vemo: red konvergence bo vsaj 2, če je

① $g(\alpha) = \alpha$,

② $g'(\alpha) = 0$.

$$g(\alpha) = \alpha - \frac{\overbrace{f(\alpha)}^{=0}}{\underbrace{f'(\alpha)}_{\neq 0}} = \alpha \quad \checkmark$$

$$g'(x) = 1 - \frac{f'(x)^2 - f(x) f''(x)}{f'(x)^2}$$

$$= \frac{f(x) f''(x)}{f'(x)^2} \Rightarrow g'(\alpha) = 0$$

$\Rightarrow$ red je vsaj 2!!

Višji red konvergence:

$$g''(x) = \frac{\left(f'(x) f''(x) + f(x) f'''(x)\right) f'(x)^2}{f'(x)^4} - $$

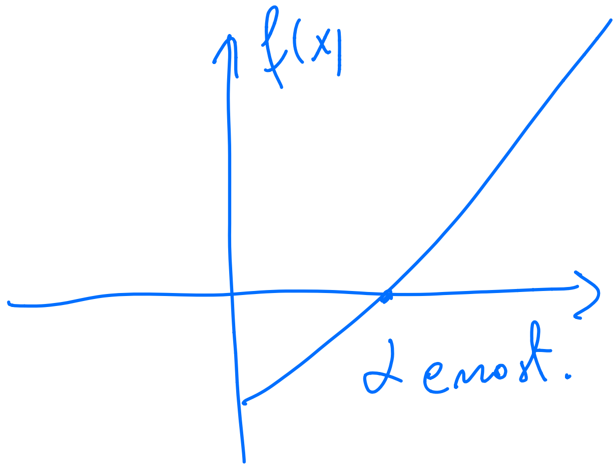
$$\frac{f(x) f''(x)\, 2 f'(x) f''(x)}{f(x)^2} \Rightarrow$$

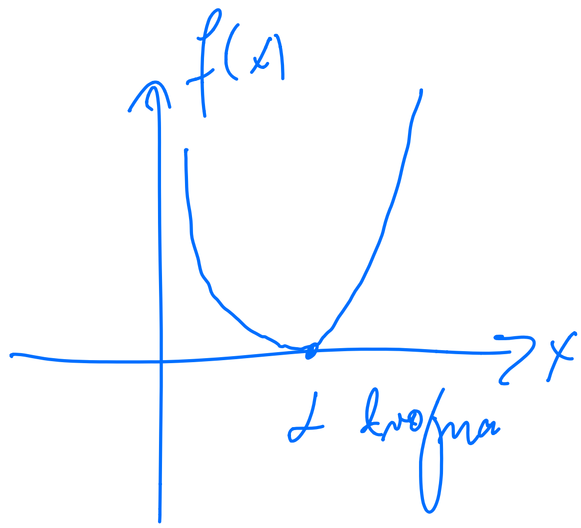
$$g''(\alpha) = \frac{\cancel{f'(\alpha)}\, f''(\alpha)\, \cancel{f'(\alpha)^3}}{\cancel{f'(\alpha)^4}} = f''(\alpha) \doteq 0$$

Če je $f''(\alpha) = 0 \Rightarrow$ red je vsaj 3.

Kaj pa če je α večkratna ničla?

Lema: Če je α večkratna ničla funkcije f, potem tangentna metoda konv. linearno.





Dokaz: Naj bo α m-kratna, $m \geq 2$:

$$f(x) = (x-\alpha)^m h(x) \; ; \; h(\alpha) \neq 0 .$$

$$f'(x) = m(x-\alpha)^{m-1} h(x) + (x-\alpha)^m h'(x)$$

$$f''(x) = m(m-1)(x-\alpha)^{m-2} h(x)$$

$$\left.\begin{array}{l} + m(x-\alpha)^{m-1} h'(x) \\ + m(x-\alpha)^{m-1} h'(x) \end{array}\right\} 2m(x-\alpha)^{m-1} h'(x)$$

$$+ (x-\alpha)^m h''(x)$$

$$g(\alpha) = \alpha - \frac{f(\alpha)}{f'(\alpha)}$$

$$= \alpha$$

$$g(x) = x - \frac{(x-\alpha)^{m} h(x)}{m\cancel{(x-\alpha)^{m-1}} h(x) + (x-\alpha)^{m} h'(x)}$$

$$g'(x) = \frac{f(x)\, f''(x)}{f'(x)^2} =$$

$$= \frac{(x-\alpha)^m h(x) \left(m(m-1)(x-\alpha)^{m-2} h(x) + 2m(x-\alpha)^{m-1} h'(x) + (x-\alpha)^m h''(x) \right)}{\left(m(x-\alpha)^{m-1} h(x) + (x-\alpha)^m h'(x) \right)^2}$$

$$= \frac{\cancel{(x-\alpha)^{2m-2}} h(x) \left(m(m-1) h(x) + 2m(x-\alpha) h'(x) + (x-\alpha)^2 h''(x) \right)}{\cancel{(x-\alpha)^{2m-2}} \left(m h(x) + (x-\alpha) h'(x) \right)^2}$$

$x=\alpha$: $\quad g'(\alpha) = \dfrac{m(m-1)}{m^2} = 1 - \dfrac{1}{m} \neq 0, \quad m \geq 2$

Ocena natančnosti približka:

Če za f na $I = [\alpha - d, \alpha + d]$, $d > 0$,

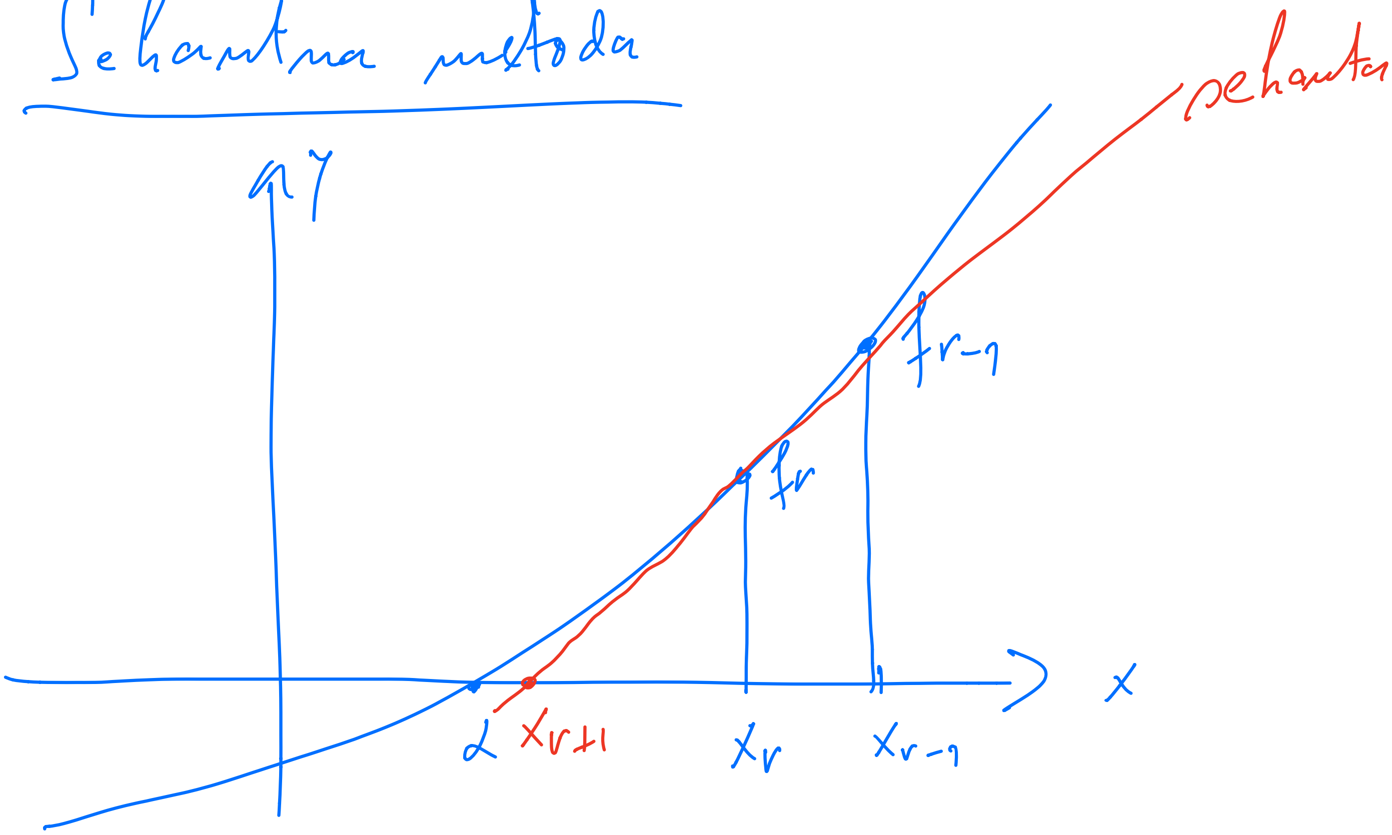
α ničla funkcije f, velja

① $|f'(x)| \geq m_1 > 0$,

② $|f''(x)| \leq M_2$,

Potem je $$|x_{r+1} - \alpha| \leq \frac{M_2}{2m_1^2} f'(x_r)|x_{r+1}-x_r|^2$$

Sekantna metoda



Velja: $$x_{r+1} = x_r - \frac{f(x_r)(x_r - x_{r-1})}{f(x_r) - f(x_{r-1})}; \quad r = 1, \dots$$

Da se preveriti: red konvergence je

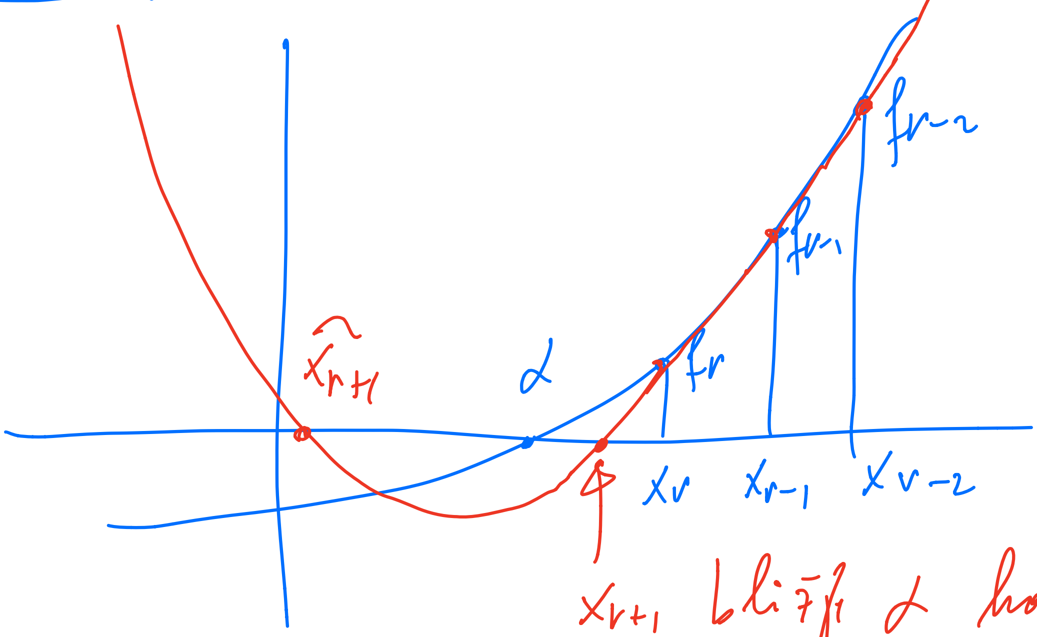
$$p = \frac{1+\sqrt{5}}{2} \doteq 1,6$$

Metoda (f, f', f''):

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} - \frac{f''(x_r)\, f^2(x_r)}{2 f'^3(x_r)}; \quad r = 0, \dots$$

Konvergenca je v splošnem kubična!

Müllerjeva metoda:



x_{r+1} bližji α kot $\tilde{x}_{r+1}$

$$p(x) = a(x-x_r)^2 + b(x-x_r) + c$$

$\vdots$

$$x_{r+1} = x_r - \frac{2c}{b + \operatorname{sign}(b)\sqrt{b^2 - 4ac}}$$

$$a = \frac{(x_{r-1}-x_r)(f_r-f_{r-1}) - (x_{r-2}-x_r)(f_{r-1}-f_r)}{(x_{r-2}-x_r)(x_{r-1}-x_r)(x_{r-2}-x_{r-1})}$$

$b = \ldots$

$c = f(x_r) = f_r$

MATLAB: fzero (Brentova metoda)

Iskanje ničel polinomov:

Iščemo ničle $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, $a_n \neq 0$.

Definiramo t.i. pridruženo matriko:

$$C_{p_n} := \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \\ \vdots & & & & \ddots & \\ & & & & 0 & 1 \\ -\frac{a_0}{a_n}, & -\frac{a_1}{a_n}, & \cdots & & , & -\frac{a_{n-1}}{a_n} \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Izrek: Lastne vrednosti matrike C_{p_n}

so vsa ničle p_n.

Dokaz: (vaje)

Matlab ničle polinoma išče na tak način: roots.

Primer: $p(x) = x^3 - 5x + 1$.

Durand - Kernerjeva metoda

Za monični polinom p st. n velja:

$$p(z) = (z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n)$$

Naj bodo $z_1, z_2, \dots, z_n$ približki za ničle $\alpha_1, \alpha_2, \dots, \alpha_n$.
Iščemo take popravke $\Delta z_1, \Delta z_2, \dots, \Delta z_n$, da bodo $z_1 + \Delta z_1, \dots, z_n + \Delta z_n$ ničle p.

Veljati mora:

$$(z-(z_1+\Delta z_1))(z-(z_2+\Delta z_2))\dots(z-(z_n+\Delta z_n))$$

$$= p(z)$$

Če slednji produkt uredimo po členih Δz_i, dobimo:

$$p(z) = \prod_{j=1}^{n} (z - z_j) - \sum_{j=1}^{n} \Delta z_j \prod_{\substack{k=1 \\ k \neq j}}^{n} (z - z_k) + \sum_{\substack{j,k=1 \\ j<k}}^{n} \Delta z_j \Delta z_k \prod_{\substack{i=1 \\ i \neq j,k}}^{n} (z - z_i) + \dots$$

Sedaj zanemarimo kvadratne in člene višje stopnje in torej Δz_i postanejo le približni popravki! Če v zgornjo enakost vstavimo $z = z_i$, sledi

$$\Delta z_i = - \frac{p(z_i)}{\prod_{\substack{k=1 \\ k \neq i}}^{n} (z_i - z_k)}$$

Durant-Kerner:

$$z_i^{(r+1)} = z_i^{(r)} - \frac{p(z_i^{(r)})}{\prod_{\substack{k=1 \\ k \neq i}}^{n} (z_i^{(r)} - z_k^{(r)})}, \quad i = 1, 2, \dots, n$$

Če se ne ničle enostavne, je konverg. kvadratična in konvergira praktično za $\forall$ začetni vektor kompleksnih števil.

Varianta:

$$z_i^{(r+1)} = z_i^{(r)} - \frac{p(z_i^{(r)})}{\prod_{k=1}^{i-1}\left(z_i^{(r)} - z_k^{(r+1)}\right)\prod_{k=i+1}^{n}\left(z_i^{(r)} - z_k^{(r)}\right)}$$

Numerično reševanje sistemov lin. enačb

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

a_{ij} in b_i so znane količine, x_i so neznanke

Matrični zapis:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$b = [b_1, b_2, \dots, b_m]^T \in \mathbb{R}^m = \mathbb{R}^{m \times 1}$$

$$x = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$$

$$\boxed{Ax = b}$$

matrični zapis sistema lin. enačb

Množenje vektorja z matriko :

$$\begin{array}{cc} m & \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \\ & n \end{array} \begin{array}{cc} n & \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ & 1 \end{array} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} \cdot x_j \\ \vdots \\ \vdots \end{bmatrix}$$

Druga možnost: $A = [a_1, a_2, \dots, a_n]$,

$a_j \in \mathbb{R}^m$ – j-ti stolpec A:

$$a_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

$x = [x_1, x_2, \ldots, x_n]^T$:

$$\boxed{Ax = \sum_{j=1}^{m} x_j a_j}$$

<u>Primer</u>: $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$Ax = \begin{bmatrix} -5 \\ 18 \end{bmatrix}$: standardno

$Ax = 1 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ 18 \end{bmatrix}$ ✓

Kdaj je sistem $Ax = b$ rešljiv?

Eden od rezultatov: $\operatorname{rang}(A) = \operatorname{rang}([A, b])$

~~Če je $\operatorname{rang}([A]) = \operatorname{rang}([A, b])$~~ $\Rightarrow$

~~b je odvisen od stolpcev $a_1, \ldots, a_n$~~

$Ax = b$ ima rešitev $\Rightarrow$

$b = \sum_{j=1}^{n} x_j a_j \Rightarrow b$ je lin. komb a_j

$\Rightarrow \operatorname{rang}(A) = \operatorname{rang}([A, b])$

In obratno: če $\operatorname{rang}(A) = \operatorname{rang}([A, b])$

$\Rightarrow b$ je lin. kombinacija $a_j \Rightarrow$

$b = \sum_{j=1}^{n} x_j a_j \Rightarrow Ax = b$ ima rešitev