Zaporedja realnih števil

$$a_n = \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \dots + \frac{1}{\sqrt{2n}}$$

navzg. neomejeno : $a_n \geq \frac{n+1}{\sqrt{n}} > \sqrt{n}$

naraščajoče :

$a_{n+1} \geq a_n$ za vse $n \in \mathbb{N}$, $n \geq 1$

$$\left.\begin{array}{l} \overbrace{\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \dots + \frac{1}{\sqrt{2n+2}}}^{a_{n+1}} \overset{?}{\geq} \\ \frac{1}{\sqrt{n}} + \dots + \frac{1}{\sqrt{2n}} \end{array}\right\} \star$$

$\star \iff$

$$\left.\frac{1}{\sqrt{2n+1}} + \frac{1}{\sqrt{2n+2}} \geq \frac{1}{\sqrt{n}}\right] \star\star$$

$$a_{n+1} = \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \dots + \frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{2n+1}} + \frac{1}{\sqrt{2n+2}}$$

$$\frac{1}{\sqrt{2n+1}} + \frac{1}{\sqrt{2n+2}} \geq \sim \geq \frac{1}{\sqrt{n}}$$

$$\frac{1}{\sqrt{2n+1}} + \frac{1}{\sqrt{2n+2}} \geq \frac{2}{\sqrt{2n+2}} \begin{matrix} :2 \\ :2 \end{matrix} =$$

$$= \frac{1}{\sqrt{\frac{n}{2}+\frac{1}{2}}} \geq \frac{1}{\sqrt{n}}$$

↳ $n \geq 1$

$$\frac{n}{2} + \frac{1}{2} \leq n \Rightarrow$$

$\star \iff \star \ \star$

$a_{n+1} \geq a_n$ za vse $n \in \mathbb{N}$

$$a_n = \sqrt[n]{n!}$$

NARAŠČAJOČE :

$$a_{n+1} \geq a_n$$

$$\sqrt[n+1]{(n+1)!} \geq \sqrt[n]{n!} \qquad \Big/ \left(\ \right)^{n \cdot (n+1)}$$

$$\Longleftrightarrow$$

$$\left[(n+1)!\right]^n \geq \left[n!\right]^{n+1} = n! \cdot (n!)^n$$

$$\parallel$$

$$(n+1)^n \cdot (n!)^n$$

$$(n+1)^n \geq n! \quad ✓$$

produkt
n faktorjev
n+1

$$\underbrace{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}_{n \text{ faktorjev}, \ < n+1}$$

Torej je zaporedje res naraščajoče.

NAVZGOR NEOMAJANO

Ne obstaja $M \in \mathbb{R}$, da je $a_n \le M$ za vsak $n \in \mathbb{N}$, $n \ge 1$

S protislovjem:

Pa recimo, da obstaja $M \in \mathbb{N}$, da je

$$a_n \le M \quad \text{za vse } n \ge 1$$

$$\sqrt[n]{n!} \le M \iff n! \le M^n$$

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n \le M \cdot \ldots \cdot M$$

$$\iff \quad | : M^n$$

$$\underbrace{\underbrace{\frac{1}{M} \cdot \frac{2}{M} \cdot \ldots \cdot \frac{M}{M}}_{c \in \mathbb{R},\ c > 0} \cdot \underbrace{\frac{M+1}{M} \cdot \ldots \cdot \frac{n-1}{M} \cdot \frac{n}{M}}_{\downarrow n \to \infty \atop \infty}}_{\downarrow \atop \infty} \le 1$$

za $n > M$

$\times$

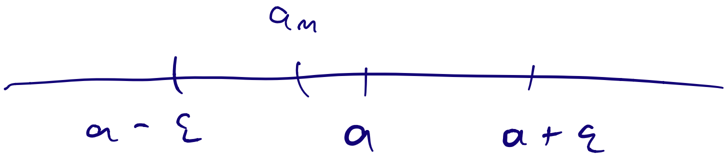
Torej, ne obstaja $M \in \mathbb{N}$, da je $a_n \leq M$ za vse $n \in \mathbb{N}$.
$(a_n)_n$ je torej res neomejeno.

Stekališča in limite

$(a_n)_n$ zaporedja realnih števil

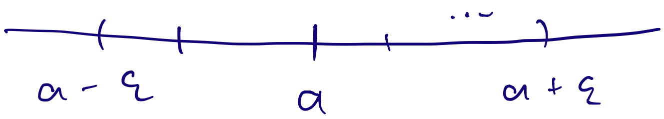
$a \in \mathbb{R}$ je stekališče, če je

za $\forall \varepsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N : |a_n - a| < \varepsilon$



$a \in \mathbb{R}$ je limita od $(a_n)_n$, če je edino stekališče zaporedja in je zaporedje omejeno.

$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N : |a_n - a| < \varepsilon$



Določi vsa stekališča zaporedja $(a_n)_n$!

- $a_n = (-1)^n + 1^n$

$2, 0, 2, 0, 2, 0, \ldots$

$$a_n = \begin{cases} 2 ; & n \text{ sod} \\ 0 ; & n \text{ lih} \end{cases}$$

Stekališči sta 0 in 2.

- $a_n = \frac{n}{n+1} \cos\left(\frac{n\pi}{2}\right)$

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 1 ; & n = 0 + 4k ; \ k \in \mathbb{N} \\ 0 ; & n \text{ lih} \\ -1 ; & n = 2 + 4k ; \ k \in \mathbb{N} \end{cases}$$

$$\frac{\overset{n}{(x+4k)} \cdot \pi}{2} = 2k\pi + \frac{x\pi}{2}$$

$0, 0, -\frac{2}{3}, 0, +\frac{4}{5}, 0, -\frac{6}{7}, 0, \frac{8}{9}, 0, -\frac{10}{11}, \ldots$

$\downarrow$ $\downarrow$ $\searrow$

$n=0$ $n=1$ $n=2$

- 0 je očitno stekališče, ker je $a_n = 0$ za vsak lih n

$$\boxed{\begin{aligned} &\frac{2}{3}, \frac{6}{7}, \ldots \longrightarrow 1 \\ &\lim_{n\to\infty} a_n = a \iff \lim_{n\to\infty} -a_n = -a \end{aligned}}$$

$$\Downarrow$$

$$-\frac{2}{3}, -\frac{6}{7}, -\frac{10}{11}, \ldots \longrightarrow -1$$

- Drugo stekališče je -1.
- $\frac{4}{5}, \frac{8}{9}, \frac{12}{13}, \ldots \longrightarrow 1$
- 1 je tretje stekališče.

Drugih stekališč ni, ker smo naše zaporedje izčrpali s podzaporedji.

Vsako realno št. je limita zaporedja racionalnih števil.

$x \in \mathbb{R}$,

$x = a.a_1 a_2 a_3 a_4 \ldots$

$q_n = a.a_1 a_2 \ldots a_n 000$

$q_n \xrightarrow{n \to \infty} x$

(q_n) je Cauchyjevo

Kdaj je zaporedje konvergentno?

- ima le eno stekališče in je omejeno
- zaporedje je naraščajoče/padajoče in navzgor/navzdol omejeno
- je Cauchyjevo
- $a_n \le b_n \le c_n$ za vse $n \in \mathbb{N}$

$\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} c_n = x \in \mathbb{R}$

"sendvič"

$\Rightarrow \lim\limits_{n \to \infty} b_n = x$

$(a_n)_n$ naj bo omejeno zaporedje
$(b_n)_n$ zaporedje z limito nič.

$(a_n b_n)_n$ ima limito nič

Obstaja $M \in \mathbb{N}$, da velja:
za vse $n \in \mathbb{N}$ velja

$$-M \le a_n \le M \quad \Big] *$$

$(a_n)_n = \{a_0, a_1, a_2, \dots\}$

Ideja : $a_n b_n$ "stisnemo" med dve zaporedji z limito nič

* pomnožimo z b_n za vsak $n \in \mathbb{N}$

$$b_n \xrightarrow{n\to\infty} 0 \Leftrightarrow |b_n| \xrightarrow{n\to\infty} 0$$

$$\begin{tikzcd}
-M|b_n| \arrow[d, "n\to\infty"] & \le & a_n|b_n| \arrow[d, "n\to\infty"] & \le & M|b_n| \arrow[d, "n\to\infty"] \\
0 & & 0 & & 0
\end{tikzcd}$$

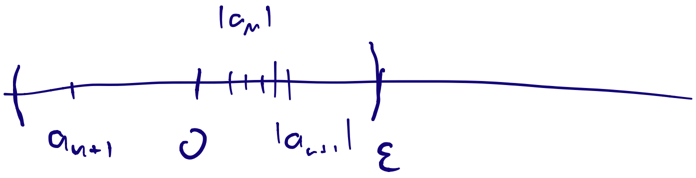
$(a_n|b_n|)_n$ ima limito nič

$\Downarrow$

$(|a_n b_n|)_n$ ima limito nič

$$\Updownarrow$$

$(a_n b_n)_n$ ima limito nič.



$(a_n b_n)_n$ ima limito nič.

$$-M \leq a_n \leq M$$

za take n, kjer je $b_n \geq 0$

$$-M b_n \leq a_n b_n \leq M b_n \longrightarrow 0$$

za take n, kjer je $b_n < 0$

$$-M b_n \geq a_n b_n \geq M b_n \longrightarrow 0$$

$\downarrow$ $\downarrow$

0 0

Izračunaj limite zaporedij!

a) $$\lim_{n \to \infty} \frac{1 + 2 + 3 + \ldots + n}{\sqrt{n^4 + n}} =$$

$$= \lim_{n \to \infty} \frac{n(n+1)}{2\sqrt{n^4 + n}} =$$

$$= \lim_{n \to \infty} \frac{n^2 + n}{2\sqrt{n^4 + n}} =$$

$$= \lim_{n \to \infty} \frac{1 + \boxed{\frac{1}{n}} \xrightarrow{n \to \infty} 0}{2\sqrt{1 + \boxed{\frac{1}{n^3}}} \xrightarrow{n \to \infty} 0} = \frac{1}{2}$$

b) $$\lim_{n \to \infty} \frac{1 + 3 + 5 + \ldots + (2n-1)}{n + 3} - n = *$$

$$1 + 3 + 5 + \ldots + (2n-1) =$$

$$= 1 + 2 + 3 + 4 + \ldots + (2n-2) + (2n-1) + 2n$$

$$- (2 + 4 + 6 + \ldots + 2n) =$$

$$= \frac{\cancel{2}n(2n+1)}{\cancel{2}} - \cancel{2}(\underbrace{1 + \dots + n}_{\frac{n(n+1)}{\cancel{2}}}) =$$

$$= 2n^2 + n - n^2 - n = n^2$$

$$* = \lim_{n\to\infty} \frac{n^2}{n+3} - n =$$

$$= \lim_{n\to\infty} \frac{n^2 - n(n+3)}{n+3} = \lim_{n\to\infty} \frac{n^2 - n^2 - 3n}{n+3} =$$

$$= \lim_{n\to\infty} \frac{-3n}{n+3} = \lim_{n\to\infty} \frac{-3}{1 + \underbrace{\frac{3}{n}}_{\downarrow\, 0}} = -3$$

$$c)\ \lim_{n\to\infty} \left(n^2\sqrt{n^2+1} - n^3\right) = \quad a, b :$$

$$(a-b)(a+b) = a^2 - b^2$$

$$= \lim_{n\to\infty} n^2 \left[\sqrt{n^2+1} - n\right] =$$

$$\hookrightarrow \cdot \frac{\sqrt{n^2+1} + n}{\sqrt{n^2+1} + n}$$

$$= \lim_{n\to\infty} n^2 \frac{n^2+1 - n^2}{\sqrt{n^2+1} + n} =$$

$$= \lim_{n\to\infty} \frac{n^2 \quad | : n^2}{\sqrt{n^2+1} + n \quad : n^2} = \infty$$

d) $$\lim_{n\to\infty} \frac{2^{n+1} + 3^{n+1} \quad : 3^{n+1}}{2^n + 3^n \quad : 3^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{\frac{2^{n+1}}{3^{n+1}} + 1}{\frac{2^n}{3^{n+1}} + \frac{1}{3}} =$$

$$= \lim_{n\to\infty} \frac{1 + \left(\frac{2}{3}\right)^{n+1} \xrightarrow{n\to\infty} 0}{\underbrace{\frac{2^{n+1}}{3^{n+1}} \cdot \frac{1}{2}}_{\substack{n\to\infty \\ \downarrow \\ 0}} + \frac{1}{3}} = \underline{\underline{3}}$$

$$\lim_{n \to \infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \dots + a_0}{b_l n^l + \dots + b_0} =$$

$a_i \in \mathbb{R}, \; a_k \neq 0,$
$b_j \in \mathbb{R}, \; b_l \neq 0$

$$= \begin{cases} \infty & ; \; k > l \; ; \quad \frac{a_k}{b_l} > 0 \\ -\infty & ; \; \text{—''—} \; ; \quad \frac{a_k}{b_l} < 0 \\ \frac{a_k}{b_l} & ; \; k = l \\ 0 & ; \; k < l \end{cases}$$

Dokaz: obe strani delimo z $n^{\max\{k, l\}}$.

$$\lim_{n \to \infty} \frac{1 + \frac{1}{3} + \frac{1}{3^2} + \dots \frac{1}{3^{n-1}}}{1 + \frac{1}{5} + \frac{1}{5^2} + \dots + \frac{1}{5^{n-1}}} =$$

$$= \lim_{n \to \infty} \frac{\frac{1}{1 - \frac{1}{3}}}{\frac{1}{1 - \frac{1}{5}}} = *$$

$1 + \frac{1}{3} +$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}$$

če sta

$\lim_{n \to \infty} a_n$, $\lim_{n \to \infty} b_n \neq 0, \infty, -\infty$

$$* \lim_{n \to \infty} \frac{\frac{4}{5}}{\frac{2}{3}} = \frac{12}{10} = \underline{\underline{\frac{6}{5}}}$$

$$a_n = \left(-\frac{\sqrt{n}}{n} + \frac{\sqrt{n+1}}{n}\right) + \left(-\frac{\sqrt{n+2}}{n} + \frac{\sqrt{n+3}}{n}\right) +$$

$$+ \left(-\frac{\sqrt{n+4}}{n} + \frac{\sqrt{n+5}}{n}\right) + \dots + \left(-\frac{\sqrt{n+2n}}{n} + \qquad\right)$$

$$\frac{\sqrt{n+2n+1}}{n}$$

Izračunaj $\lim\limits_{n\to\infty} a_n$!

$$\left(-\frac{\sqrt{n+k}}{n} + \frac{\sqrt{n+k+1}}{n} = \frac{\sqrt{n+k+1}-\sqrt{n+k}}{n} = \right.$$

$$\cdot \frac{(\sqrt{n+k+1}+\sqrt{n+k})}{(\sqrt{n+k+1}+\sqrt{n+k})}$$

$k = 0, 2, 4, \dots, 2n$

$$= \frac{1}{n(\sqrt{n+k+1}+\sqrt{n+k})}$$

$k=0$

$k=2$

$$a_n = \underset{\downarrow \atop 0}{\frac{1}{n}} \left[\underset{\downarrow \atop 0}{\frac{1}{\sqrt{n+1}+\sqrt{n}}} + \underset{\downarrow \atop 0}{\frac{1}{\sqrt{n+3}+\sqrt{n+2}}} + \dots + \underset{k = 2n \qquad \downarrow \atop \qquad\qquad 0}{\frac{1}{\sqrt{3n+1}+\sqrt{3n}}} \right]$$

$(*)\ a_n \longrightarrow 0$

$$b_n = \underbrace{\overset{\to 0}{\frac{1}{n}} + \overset{\to 0}{\frac{1}{n}} + \dots + \overset{\to 0}{\frac{1}{n}}}_{n\text{-fach}}$$

$= 1$

$b_n \not\longrightarrow 0$

$$\frac{1}{\sqrt{n+k+1}+\sqrt{n+k}} \leq \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$; \quad k = 0, 2, 4, \dots$

$$\sqrt{n+k+1} \geq \sqrt{n+1}$$

$$\sqrt{n+k} \geq \sqrt{n}$$

$$\Rightarrow \quad a_n \leq \frac{1}{n}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\cdot(n+1)\right) =$$

$$= \underset{\substack{\downarrow \\ 1}}{\frac{n+1}{n}} \cdot \underset{\substack{\downarrow \\ 0}}{\frac{1}{\sqrt{n+1}+\sqrt{n}}} =: b_n$$

$$a_n \leq \underset{\substack{\downarrow \\ 0}}{b_n}$$

Ker je $a_n \geq 0$ za vse $n \in \mathbb{N}, n \geq 1$

$$\underset{\substack{\downarrow \\ 0}}{0} \leq a_n \leq \underset{\substack{\downarrow \\ 0}}{b_n}$$

$$a_n \xrightarrow{n \to \infty} 0 \quad \text{za vse } n \in \mathbb{N}$$

$$a_n = \frac{(3n)!}{n!\,(2n)!\,2^{3n}}$$

Dokaži, da je zaporedje (od nekega N naprej) padajoče.

· Izračunaj $\lim_{n\to\infty} a_n$

Padajoče:

$$a_{n+1} \leq a_n$$

$$\frac{a_{n+1}}{a_n} \leq 1$$

$$\frac{(3(n+1))! \cdot n!\,(2n)!\,2^{3n}}{(n+1)!\,(2(n+1))!\,2^{(3(n+1))} \cdot (3n)!} =$$

$$= \frac{(3n+3)!\,\underline{n!}(2n)!}{\underline{(n+1)!}\,(2n+2)! \cdot 2^3\,(3n)!} =$$

$$= \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1) \cdot 2^3} =$$

$$= \frac{3}{16} \frac{(3n+2)(3n+1)}{(2n+1)(n+1)} =$$

$$\frac{3n+3}{2n+2} = \frac{3(n+1)}{2(n+1)}$$

$$= \frac{3(9n^2+9n+2)}{16(2n^2+3n+1)} \xrightarrow{n \to \infty} \frac{27}{32}$$

od neke naprej: $\frac{a_{n+1}}{a_n} \leq 1$

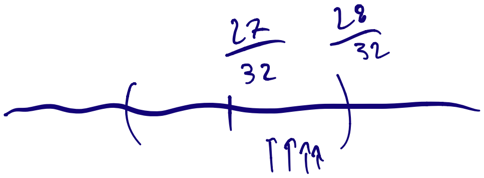
↳ za vse $n \geq N$ za neki $N \in \mathbb{N}$

Obstaja $N \in \mathbb{N}$, da je $\frac{a_{n+1}}{a_n} \leq 1$

$\Leftrightarrow a_{n+1} \leq a_n$ za vse $n \geq N$

Torej je zaporedje od neke naprej res padajoče.

Izračunaj $\lim_{n\to\infty} a_n$



Namig: $\frac{a_{n+1}}{a_n} \xrightarrow{n\to\infty} \frac{27}{32}$

Torej: $\frac{a_{n+1}}{a_n} \le \frac{28}{32}$ za vse n od nekega $N' \in \mathbb{N}$ naprej. Vzamemo $n \ge N'$

$$a_{n+1} \le \frac{28}{32} a_n$$

$$a_{n+2} \le \frac{28}{32} a_{n+1} \le \left(\frac{28}{32}\right)^2 a_n$$

$$a_{n+3} \le \left(\frac{28}{32}\right)^3 a_n$$

$$\begin{tikzcd} 0 \le a_{n+k} \arrow[d, "k\to\infty"] & \le & \left(\frac{28}{32}\right)^k \underset{\mathbb{R}}{\underset{\cap}{a_n}} \arrow[d, "k\to\infty"] \\ 0 & & 0 \end{tikzcd}$$

(fiksirali smo n)

Torej je tudi $\lim\limits_{n \to \infty} a_n = 0$.

Izračunaj $\limsup\limits_{n \to \infty} a_n$ in $\liminf\limits_{n \to \infty} a_n$,

kjer je

$$a_n = \begin{cases} \frac{1}{n+1} \; ; & n \text{ sod} \\ \frac{n}{n+1} \; ; & n \text{ lih} \end{cases}$$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5$

$1, \; \frac{1}{2}, \; \frac{1}{3}, \; \frac{3}{4}, \; \frac{1}{5}, \; \frac{5}{6}$

$$\lim_{n \to \infty} a_{n \cdot 2} = \lim_{n \to \infty} \frac{1}{2n+1} = 0$$

$$\lim_{n \to \infty} a_{n \cdot 2 + 1} = \lim_{n \to \infty} \frac{2n+1}{2n+2} = 1$$

$$\limsup a_n = 1, \quad \liminf a_n = 0$$