13. The Logistic Differential Equation

Suppose that P(t) describes the quantity of a population at time t. For example, P(t) could be the number of milligrams of bacteria in a particular beaker for a biology experiment, or P(t) could be the number of people in a particular country at a time t. A model of population growth tells plausible rules for how such a population changes over time. The simplest model of population growth is the **exponential model**, which assumes that there is a constant parameter r, called the **growth parameter**, such that

$$P'(t) = r\,P(t)$$

holds for all time t. This differential equation itself might be called the **exponential differential equation**, because its solution is

$$P(t) = P_0\,e^{rt}$$

where $P_0 = P(0)$ is the initial population.

One noticeable feature of the exponential model is that, when r is positive, the population always grows larger and larger without any finite limit, as is seen in Figure 1. This kind of growth may be a good model for a new population of bacteria in a beaker, but it does not hold for a long time. It is easy to see that the equation would imply a population of bacteria that ultimately outgrew the beaker and even outgrew the planet earth, since the mass of the bacteria would ultimately exceed the mass of the earth. Such a model is therefore absurd to model a system for long periods of time. The fundamental difficulty is that the exponential differential equation ignores the fact that there are limits to resources needed for the population to grow. It ignores the needs for food, oxygen, and space; and it ignores the accumulation of waste products that inevitably arise.

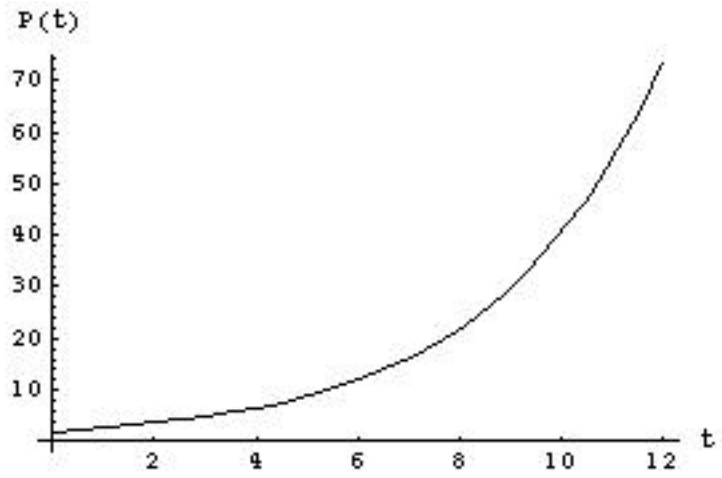

Figure1 : $P(t) = 2\,e^{0.3t}$. This example of the exponential model shows a population growing always faster without any bound.

An alternative model was proposed by Verhulst in 1836 to allow for the fact that there are limits to growth in all known biological systems. He proposed what is now called the **logistic differential equation.** The equation involves two positive parameters. The first parameter r is again called the **growth parameter** and plays a role similar to that of r in the exponential differential equation. The second parameter K is called the **carrying capacity**. The logistic differential equation is written

$$P'(t) = r\,P(t)\,[1 - P(t)/K].$$

Equivalently, in terms of the d notation, the logistic differential equation is

$$dP/dt = r\,P\,[1 - P / K]$$

Note that when P(t) is very small, then P(t)/K is close to 0, so the entire factor [1-P(t)/K] is close to 1 and the equation itself is then close to P'(t) = r P(t); we then expect that the population grows approximately at an exponential rate when the population is small. On the other hand, if P(t) gets to be near K, then P(t)/K will be approximately 1, so [1-P(t)/K] will be approximately 0, and the logistic differential equation will then say approximately P'(t) = r P(t) 0 = 0. The growth rate will be essentially 0, so the population will not grow significantly more.

To solve the logistic differential equation, we separate variables:
$dP/dt = r P [1 - P / K]$
$dP = r P [1 - P / K] dt$

$$\frac{dP}{[P (1 - P / K)]} = r\ dt$$

$$\int \frac{dP}{[P (1 - P / K)]} = \int r\ dt$$

Since the left integral is quite difficult, the remainder of the details will be left to the end of this section. There is an arbitrary constant C in the general solution. When the fact that $P(0) = P_0$ is utilized, we obtain that

$$P(t) = \frac{P_0 K}{P_0 + (K-P_0) e^{-rt}}$$

Theorem 1. The solution of the logistic differential equation is

$$P(t) = \frac{P_0 K}{P_0 + (K-P_0) e^{-rt}}$$

where $P_0 = P(0)$ is the **initial population**.

This formula is the **logistic formula**. It tells the equation for the **logistic curve**. The derivation of the formula will be given at the end of this section.

The logistic curve gives a much better general formula for population growth over a long period of time than does exponential growth.

Example. A population of bacteria grows according to the differential equation
$dP/dt = 0.03 P (1 - P/2000)$
When t = 0, the population is 200 g.
Find the population P at time t.

Solution. We recognize the logistic differential equation with r = 0.03 and K = 2000. We are given $P_0 = 200$. Hence by Theorem 1,

$$P(t) = \frac{(200)(2000)}{(200) + (2000-200) e^{-0.03t}}$$

$$P(t) = \frac{400{,}000}{200 + 1800 e^{-0.03 t}}$$

Example. A population of bacteria grows according to the differential equation

$$dP/dt = 0.05\ P\ (1 - 0.001\ P)$$

When t = 0, the population is 300 g.
Find the population P at time t.

Solution. We recognize the logistic differential equation with r = 0.05. To find K note that the part of the general differential equation involving K is

$$(1 - P/K)$$

while for the problem at hand the part is

$$(1 - 0.001\ P)$$

Hence $1/K = 0.001$
so K = 1000.
We are given $P_0 = 300$. Hence by Theorem 1,

$$P(t) = \frac{(300)\ (1000)}{(300) + (1000-300)\ e^{-0.05t}}$$

$$P(t) = \frac{300{,}000}{300 + 700\ e^{-0.05\ t}}$$

Since the numbers are messy, we might simplify by dividing numerator and denominator by 100. We then get

$$P(t) = \frac{3000}{3 + 7\ e^{-0.05\ t}}$$

Example. A population of bacteria grows logistically. Suppose the initial population is 3 mg of bacteria, the carrying capacity is 100 mg, and the growth parameter is 0.2 $hour^{-1}$.
(a) Find the differential equation satisfied by the population.
(b) Find the population at all times.
(c) When will the population reach 90 mg?
(d) When will the population reach 200 mg?

Solution. (a) The logistic differential equation is

$$P'(t) = r\ P(t)\ [1 - P(t)/K]$$

Since P0 = 3, K = 100, r = 0.2, the equation is

$$P'(t) = 0.2\ P(t)\ [1 - P(t) / 100]$$
$$P'(t) = 0.2\ P(t)\ [1 - 0.01\ P(t)]$$

(b) From Theorem 1 we know that

$$P(t) = \frac{P_0\ K}{P_0 + (K-P_0)\ e^{-rt}}$$

But we are given that $P_0 = 3$ mg of bacteria, K = 100 mg, and r = 0.2 $hour^{-1}$.
Plugging in these values we see

$$P(t) = \frac{(3)(100)}{3 + (100-3)\ e^{-0.2t}} = \frac{300}{3 + 97\ e^{-0.2t}}$$

(c) The population will reach 20 mg when P(t) = 20.

$$\frac{300}{3 + 97\, e^{-0.2t}} = 20$$

$3 + 97\, e^{-0.2t} = 300/20 = 15$
$97\, e^{-0.2t} = 12$
$e^{-0.2t} = 12/97$
$-0.2t = \ln(12/97)$
$t = [\ln(12/97)]/(-0.2) = 10.45$ hours

(d) The population will reach 200 mg when P(t) = 200.

$$\frac{300}{3 + 97\, e^{-0.2t}} = 200$$

$3 + 97\, e^{-0.2t} = 300/200 = 1.5$
$97\, e^{-0.2t} = 1.5 - 3 = -1.5$
$e^{-0.2t} = -1.5/97$
$-0.2t = \ln(-1.5/97)$
But this is meaningless. There is no such t since one cannot take the logarithm of a negative number. Hence the population never reaches 200 mg.

Figure 2 shows this logistic curve. Note that it has roughly the shape of an elongated S (and it is in fact sometimes called the **S-shaped curve**). The population initially grows slowly but steadily. Then the growth speeds up and the curve moves more steeply upward. As the population gets closer to the carrying capacity K = 100, the growth slows and the curve gets more horizontal again. In fact the population never appears to reach the carrying capacity, but instead seems to approach it as an asymptote. Hence it is clear from Figure 2 why we never find a t such that P(t) = 200.

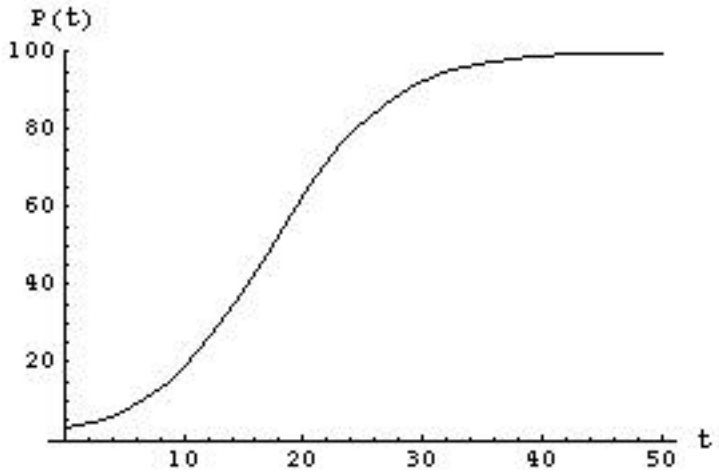

Figure 2: The logistic curve when $P_0 = 3$, K = 100, r = 0.2. This is an example of the famous S-shaped curve. It shows limits to population growth.

There are many different ways to express the formula for P(t). In the preceding example, we had

$$P(t) = = \frac{300}{3 + 97\, e^{-0.2t}}$$

We could multiply numerator and denominator by any number to find an equivalent formula. For example, we could multiply numerator and denominator by (1/3):

$$P(t) = = \frac{(300)\quad (1/3)}{(3 + 97\, e^{-0.2t})\ (1/3)}$$

$$P(t) = = \frac{100}{1 + (97/3)\, e^{-0.2t}}$$

$$P(t) = = \frac{100}{1 + 32.33\, e^{-0.2t}}$$

This last expression looks nice since it has 1 in the first term of the denominator.

Alternatively, we might like 1 in the numerator. Hence

$$P(t) = = \frac{(300)\quad (1/300)}{(3 + 97\, e^{-0.2t})\ (1/300)}$$

$$P(t) = = \frac{1}{3/300 + (97/300)\, e^{-0.2t}}$$

$$P(t) = = \frac{1}{0.01 + 0.3233\, e^{-0.2t}}$$

Thus when someone gives us a formula for the logistic curve, we must remember that they might have "simplified" it in such a manner. It is then not quite so easy to find K and P_0.

The identification of K is made easy in general because, in fact, it is true that as t goes to infinity, P(t) will approach K as an asymptote. This is seen more exactly using limits:

Example. Find

$$\lim_{t \to \infty} \frac{550}{10 + 20\, e^{-0.1\, t}}$$

Solution. When we write $t \to \infty$ we mean that t gets very large without bound. Note that

$$e^{-0.1t} = 1 / e^{0.1t}$$

As $t \to \infty$ we see that the exponent in the denominator gets huge. So the whole denominator gets huge (think of it as like $2^{1,000,000}$) so the fraction goes to 0. Hence

$$\lim_{t \to \infty} e^{-0.1t} = 0$$

and

$$\frac{550}{}$$

$$= \lim_{t \to \infty} \frac{550}{10 + 20\, e^{-0.1\, t}}$$

$$= \frac{550}{10 + 20\,(0)}$$

$= 550/10 = 55$

The same technique is used in the following theorem:

Theorem 2. If P(t) is the logistic formula and K is the carrying capacity, then $\lim_{t \to \infty} P(t) = K$.

Proof. Note that $e^{-rt} = 1/ e^{rt}$. Since $r>0$, as $t \to \infty$ we see that e^{rt} also goes to infinity, so e^{-rt} goes to 0. Hence

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{P_0 K}{P_0 + (K-P_0)\, e^{-rt}} = \lim_{t \to \infty} \frac{P_0 K}{P_0} = K$$

Example. Suppose that a particular population of bacteria obeys the logistic formula

$$P(t) = \frac{2000}{2 + 3\, e^{-0.3t}}$$

where P is in mg and t is in hours. Find K, r, and P_0.

Solution. First we find P_0:

$P_0 = P(0) = 2000/(2+3e^0) = 2000/(2+3) = 2000/5 = 400$ mg.

To compute K, by Theorem 2 we know that $K = \lim_{t \to \infty} P(t)$.
But as t gets arbitrarily large, $e^{-0.3t} = 1 / e^{0.3t}$ goes to 0.
Hence $\lim_{t \to \infty} P(t) = 2000/(2 + 3(0)) = 2000/2 = 1000$.
Since P is in mg, K is also in mg, so K = 1000 mg.

The numerical value of r is 0.3 since -r is the coefficient of t in the exponent of e. The exponent of e should be dimensionless, so rt should have no units. Since t is in hours, it follows that the units of r should be 1/hr. Hence $r = 0.3\ hr^{-1}$.

Thus the natural formula way in this problem would have been to find

$$P(t) = \frac{P_0 K}{P_0 + (K-P_0)\, e^{-rt}}$$

$$P(t) = \frac{(400)\,(1000)}{400 + (600)\, e^{-0.3t}}$$

$$P(t) = \frac{400{,}000}{400 + 600\, e^{-0.3t}}$$

The original formula had been simplified to make the numbers smaller.

Example. Suppose that a particular population of bacteria obeys the logistic formula

$$P(t) = \frac{2000}{1.5 + 3.7\, e^{-0.2t}}$$

where P is in mg and t is in hours.
(a) Find K, r, and P_0.
(b) Find the differential equation that P(t) solves.
Solution. (a) r = 0.2 from the exponential form.
$P_0 = P(0) = 2000/ (1.5 + 3.7\, e^{0}) = 2000 / (1.5 + 3.7) = 384.6$ mg

$$K = \lim_{t \to \infty} P(t)$$

$$= \lim_{t \to \infty} \frac{2000}{1.5 + 3.7\, e^{-0.2t}}$$

$$= \frac{2000}{1.5 + 3.7\,(0)}$$ since $e^{-0.2t}$ goes to 0 as t gets very large

= 2000/1.5 = 1333 mg
The carrying capacity is 1333 mg.

(b) The differential equation is
dP/dt = r P [1 - P/K].
Hence
dP/dt = 0.2 P [1 - P/1333].

The maximal rate of growth for the logistic formula

Note that according to the logistic differential equation we have P'(t) > 0 for populations P(t) satisfying 0 < P(t) < K. This is because, for such P(t), K - P(t) > 0, so dividing by K we have (1 - P(t)/K) > 0. But P'(t) = r P(t) (1-P(t)/K) so P'(t) > 0 as well. This implies that the logistic curve is always increasing, as occurred in the example in Figure 2.

We would like to find out when the rate of growth is maximal. This will tell when the population is growing the fastest. To do this, recall that the rate of growth is P' = dP/dt, and we know that

dP/dt = r P (1-P/K)

Hence we wish to find out when

r P (1-P/K)

is maximal.

To do this, we take its derivative. More explicitly, let
F(P) = r P (1-P/K).
Then $F(P) = rP - (r/K)\, P^2$
so F '(P) = r - 2(r/K)P
The maximum occurs with F '(P) = 0, hence

r - 2 (r/K)P = 0

$- 2 (r/K)P = -r$
$-(2/K) P = -1$
$P = K/2$

Theorem 3. The maximal rate of growth for the logistic curve occurs when
$P = K/2$.

In words, the growth rate peaks and starts to slow when the population reaches half the carrying capacity.

It is also useful to know the time when the maximal rate of growth occurs. This is found by solving the equation
$P(t) = K/2$
since it is the time when the population is half the carrying capacity.

Example. Suppose that a particular population of bacteria obeys the logistic formula

$$P(t) = \frac{3000}{1.5 + 3.5\, e^{-0.2t}}$$

where P is in mg and t is in hours.
(a) Tell the population when it is growing fastest.
(b) Tell when the population is growing fastest.
(c) Tell the maximal rate of growth for the population.

Solution. (a) The population grows fastest when $P = K/2$. Hence we need to find K.

$$K = \lim_{t \to \infty} P(t)$$

$$= \lim_{t \to \infty} \frac{3000}{1.5 + 3.5\, e^{-0.2t}}$$

$$= 3000/1.5 = 2000 \text{ mg}$$

Hence the fastest rate of growth occurs when P = 1000 mg

(b) We need to find t such that P(t) = 1000.

$$\frac{3000}{1.5 + 3.5\, e^{-0.2t}} = 1000$$

$1.5 + 3.5\, e^{-0.2t} = 3000/1000 = 3$
$3.5\, e^{-0.2t} = 1.5$
$e^{-0.2t} = 1.5 / 3.5$
$-0.2\, t = \ln(1.5/3.5)$
$t = [\ln(1.5/35)]/(-0.2) = 4.236$ hours

(c) The rate of growth of the population is P '(t). The maximal rate of growth occurs when t = 4.236 hours. Hence we need P '(4.236).

The most direct method is just to differentiate:
By the quotient rule, P '(t)

$$= \frac{(1.5 + 3.5\, e^{-0.2t})(0) - 3000\, D_t[1.5 + 3.5\, e^{-0.2t}]}{[1.5 + 3.5\, e^{-0.2t}]^2}$$

$$= \frac{-3000 (3.5)\, e^{-0.2t} (-0.2)}{[1.5 + 3.5\, e^{-0.2t}]^2}$$

Hence P '(4.236)

$$= \frac{-3000 (3.5)\, e^{-0.2(4.236)} (-0.2)}{[1.5 + 3.5\, e^{-0.2(4.236)}]^2}$$

= 100 mg/hour

A much simpler method, however, is to utilize the differential equation:
P'(t) = r P(t) [1 - P(t)/K]
Hence
P'(t) = 0.2 P(t) [1 - P(t)/2000]
P'(4.236) = 0.2 P(4.236) [1 - P(4.236)/2000]
= 0.2 (1000) [1 - (1000)/2000]
= 100 mg/hour

Application to epidemics

The logistic curve is often a much better model than the exponential curve for the growth of populations when the environment poses limitations on growth. It turns out that another use of the logistic curve is to describe the spread of an epidemic. This is true because an epidemic corresponds to the population growth of a certain disease organism.

Example. Suppose that t weeks after the start of an epidemic in a certain community, the number P(t) of people who have caught the disease is given by the logistic curve

$$P(t) = \frac{1500}{5 + 295\, e^{-0.9t}}$$

(a) How many people had the disease when the epidemic began?
(b) Approximately how many people in total will get the disease?
(c) When was the disease spreading most rapidly?
(d) How fast was the disease spreading at the peak of the epidemic?
(e) When did the spread of the disease start to slow down?
(f) What is the differential equation for P(t)?
(g) By the time 200 people had had the disease, what was the rate at which the disease was spreading?

Solution. (a) When the epidemic began, there were P(0) = 1500/(5+295(1))= 5 people ill.
(b) The number that will get the disease is the carrying capacity K. As t goes to infinity, $e^{-0.9t}$ goes to 0, so K = 1500/(5 + 295(0)) = 300. We expect a total of 300 people to get ill during the epidemic.
(c) The disease was spreading most rapidly at the inflection point. From Theorem 3, this occurred when t = $\ln[(K-P_0)/P_0] / r = \ln[(300-5)/5]/0.9 = 4.53$. Hence the epidemic will peak 4.53 weeks after the beginning of the epidemic.

(d) From (c), the peak of the epidemic occurred when t = 4.53. The rate of spreading is P'(t), so the answer will be P'(4.53). But

$$P'(t) = \frac{398250\, e^{-0.9t}}{(5+295\, e^{-0.9\, t})^2}$$

so P'(4.53) = 67.5 people/week. At the peak of the epidemic, 67.5 new people get the disease per week.

(e) Once the epidemic reaches its peak, the spread of the disease will begin to slow down. Hence the spread of the disease starts to slow 4.53 weeks after the beginning of the epidemic.

(f) P'(t) = r P(t) [1 - P(t)/K] = 0.9 P(t) [1 - P(t)/300]

(g) When P = 200, it follows that

P' = 0.9 (200) [1- 200/300] = 60 people/week.

Derivation of the logistic formula

We wish to solve the logistic differential equation

P'(t) = r P(t) [1 - P(t)/K]

Rewrite it as

dP/ dt = r P (1 - P/K)

The equation is separable, so rewrite it again as

$$\frac{dP}{P\,(1 - P/K)} = r\, dt$$

We now wish to integrate both sides. The left side requires the use of the trick of partial fractions: We guess that there are constants A and B such that

$$\frac{1}{P\,(1- P/K)} = \frac{A}{P} + \frac{B}{(1-P/K)}$$

Clear the fractions:

1 = A (1-P/K) + B P

This should be identically true, hence true when P = 0 and when P = K. When P=0 we obtain

1 = A (1-0) + B 0

so 1 = A. When P = K we obtain

1 = A (1-K/K) + B K

so 1 = BK and B = 1/K.

Hence

$$\int \frac{1\ \ dP}{P\,(1- P/K)} = \int \frac{1\ dP}{P} + \int \frac{(1/K)\ \ dP}{(1-P/K)}$$

The right hand integral can be done by the substitution u = (1-P/K) whence

du = -(1/K) dP

Hence

$$\int \frac{1\ \ dP}{P\,(1- P/K)} = \ln(P) + \int \frac{-\ du}{u}$$

$$= \ln(P) - \ln(1-P/K)\ .$$

But $\int r\, dt = rt + C$ where C is a constant of integration.

Since $\int \frac{dP}{P\,(1 - P/K)} = \int r\, dt$

we obtain

$$\ln(P) - \ln(1-P/K) = rt + C.$$

To simplify this, we take e to the power given by either side.

Hence $e^{[\ln(P) - \ln(1-P/K)]} = e^{[r\,t + C]}$

$e^{[\ln(P)]} / e^{\ln(1-P/K)]} = e^{C} e^{r\,t}$

$P/(1-P/K) = e^{C}\, e^{rt}$

Write $A = e^{C}$. Now

$$P/(1-P/K) = A\, e^{rt}$$

We need to solve this equation for P. Multiply by (1-P/K):

$$P = A\, e^{rt} (1-P/K)$$

Move all terms with P to the left:

$$P + (A\, e^{rt} / K)P = A\, e^{rt}$$

Factor: $P\,[\,1 + A\, e^{rt} / K] = A\, e^{rt}$

Hence

$$P = \frac{A\, e^{rt}}{1 + A\, e^{rt} / K}$$

Multiply numerator and denominator by Ke^{-rt}

$$P = \frac{A\, K}{K\, e^{-rt} + A\, e^{rt}\, e^{-rt}}$$

$$P = \frac{A\, K}{K\, e^{-rt} + A}$$

But $P(0) = P_0$, so

$$P_0 = \frac{A\, K}{K + A}$$

Hence $P_0\,(K + A) = AK$

$P_0\, K + P_0\, A = A\, K$

$P_0\, K = A\, K - A\, P_0$

$P_0\, K = A\,(K - P_0)$

$$A = \frac{P_0\, K}{K - P_0} = (P_0\, K)/\,(K - P_0)$$

Now plugging this back into the expression for P we obtain

$$P = \frac{A\, K}{K\, e^{-rt} + A}$$

$$P = \frac{[(P_0\, K)/\,(K - P_0)]\, K}{K\, e^{-rt} + (P_0\, K)/\,(K - P_0)}$$

Multiply numerator and denominator by $(K-P_0)/K$. Then

$$P = \frac{(P_0\, K)}{(K - P_0)\, e^{-rt} + P_0}$$

which is equivalent to the logistic formula.