$\limsup_{n\to\infty} a_n$ ... največje stekališče zaporedja oz. ∞, če je zaporedje navzgor neomejeno

$\liminf_{n\to\infty} a_n$ najmanjše stekališče zaporedja oz. $-\infty$, če je zaporedje navzdol neomejeno

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Poišči $\limsup_{n\to\infty}$ in $\liminf_{n\to\infty}$ zaporedja $(a_n)_n$.

$$a_n = n^{\sin\left(\frac{n\pi}{2}\right)}$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & ; \ n = 2k \ ; \ k\in\mathbb{Z} \\ 1 & ; \ n = 1+4k \ ; \ k\in\mathbb{Z} \\ -1 & ; \ n = -1+4k \ ; \ k\in\mathbb{Z} \end{cases}$$

$n = 2k \Rightarrow \sin\left(\frac{n\pi}{2}\right) = 0 \Rightarrow a_n = n^0 = 1$

$n = 1+4k \Rightarrow \sin\left(\frac{n\pi}{2}\right) = 1 \Rightarrow a_n = n$

$n = -1+4k \Rightarrow \sin\left(\frac{n\pi}{2}\right) = -1 \Rightarrow a_n = \frac{1}{n}$

$a_2, a_4, a_6, \dots \longrightarrow 1$

$a_1, a_5, a_9, \dots \longrightarrow \infty \quad \Rightarrow \quad \limsup_{n\to\infty} a_n = \infty$

$a_3, a_7, a_{11}, \dots \longrightarrow 0 \qquad\qquad \liminf_{n\to\infty} a_n = 0$

$(a_n)_n, (b_n)_n$ omejeni zaporedji

$$\left| \overbrace{\limsup_{n\to\infty} (a_n + b_n)}^{C} \leq \overbrace{\limsup_{n\to\infty} a_n}^{A} + \overbrace{\limsup_{n\to\infty} b_n}^{B} \right|$$

- - - - - - - - - - - - - - - -

$A := \limsup_{n\to\infty} a_n$, $B := \limsup_{n\to\infty} b_n$, $C := \limsup_{n\to\infty} (a_n + b_n)$

$C > A + B$

Torej obstaja podzaporedje

$(a_{n_k} + b_{n_k})_k \subseteq (a_n + b_n)_n$,

ki konvergira proti C.

Ideja: $(a_{n_k})_k \supseteq (a_{n_{k'}})_{k'}$

$\hookrightarrow$ podzaporedje, ki konvergira proti nekemu št., strogo večjem od A

$$(b_{n_k})_k \supseteq (b_{n_{k'}})_{k'}$$

$\hookrightarrow$ podzaporedje, ki konvergira proti št., večjem od B

če ne obstaja nobeno podzaporedje $(a_{n_k})_k$, ki bi konvergiralo proti nebemu $A' > A$, potem bo za vsak $\varepsilon > 0$ $\exists K$, da bo $|a_{n_k} - A| < \frac{\varepsilon}{2}$ za vse $k \geq K$ ali pa bo $a_{n_k} < A$

če ne obstaja nobeno podzaporedje $(b_{n_k})_k$, ki bi konvergiralo proti nebemu $B' > B$, potem bo za vsak $\varepsilon > 0$ $\exists K$, da bo $|b_{n_k} - B| < \frac{\varepsilon}{2}$ za vse $k \geq K$ ali pa bo $b_{n_k} < B$.

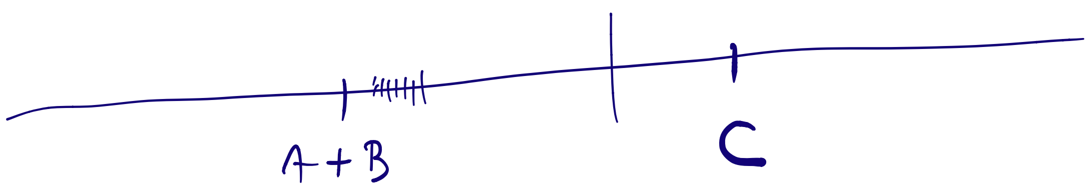
(upoštevamo, da sta zaporedji $(a_n)_n$ in $(b_n)_n$ omejeni)

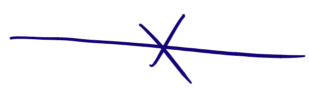
Potem velja za vsak $k \geq K$:

$$\left|\left(a_{n_k} + b_{n_k}\right) - (A+B)\right| < \varepsilon$$

ali pa bo $a_{n_k} + b_{n_k} < A + B$

Ker je $C > A + B$, bo od nekega k naprej $a_{n_k} + b_{n_k} < C - \frac{C-(A+B)}{2}$





C je limita $\left(a_{n_k} + b_{n_k}\right)$

$$\Rightarrow \limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

$$\limsup_{n\to\infty}(a_n+b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \;?$$

Dokaži da lahko velja stroga neenakost s proti primerom.

Ideja: najdemo konkretni zaporedji $(a_n)_n$ in $(b_n)_n$, za kateri velja

$$(a_n)_n = \{1, -1, 1, -1, 1, -1, \dots\}$$

$$(b_n)_n = \{-1, 1, -1, 1, -1, \dots\}$$

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$$

$$a_n + b_n = 0 \quad \text{za vsak } n \in \mathbb{N}$$

$$\limsup_{n\to\infty}(a_n+b_n) = 0$$

Recimo, da eno od zaporedij konvergira.
(BŠS : $(a_n)_n$ konvergentno)

↳ "brez škode za splošnost"

$$\limsup_{n\to\infty} a_n + b_n = \underset{\substack{\| \\ \lim\limits_{n\to\infty} a_n}}{\limsup_{n\to\infty} a_n} + \limsup_{n\to\infty} b_n$$

če vemo:

$$\underset{C}{\limsup_{n\to\infty} a_n + b_n} \not> \underset{A}{\limsup_{n\to\infty} a_n} + \underset{B}{\limsup_{n\to\infty} b_n}$$

$$\limsup_{n\to\infty} b_n =: B$$

B je (najvišje) stekališče $(b_n)_n$, torej obstaja podzaporedje $(b_{n_k})_k \subseteq (b_n)_n$, ki konvergira proti B.

Kako najdemo podzaporedje $(a_n+b_n)_n$, ki bo konverg. proti $A+B$?

Kam konvergira $(a_{n_k})_k$?

Proti A.

$(a_n+b_n)_n = \{a_1+b_1, a_2+b_2, \dots\}$ A $\qquad a_n + b_{n_k}$

$\not\Rightarrow$

Sklep: $\lim\limits_{k\to\infty} a_{n_k} + b_{n_k} = A+B$

$A+B$ je stekališče $(a_n+b_n)_n$

$$\Rightarrow \limsup_{n\to\infty} a_n+b_n \geq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

$$\not>$$

$$\Rightarrow \limsup_{n\to\infty} a_n+b_n = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

$1, 1, 2, 1, 3, 1, 4, 1, \dots \not\to \infty$

Izračunaj limite zaporedij:

- $\lim\limits_{n\to\infty}\left(1+\frac{3}{n}\right)^{n} = \lim\limits_{n\to\infty}\left(1+\frac{1}{\frac{n}{3}}\right)^{n} =$

$\left(1+\frac{1}{n}\right)^{n} \longrightarrow e$

$= \lim\limits_{n\to\infty}\left(1+\frac{1}{\frac{n}{3}}\right)^{\frac{n}{3}\cdot 3} = \left[\lim\limits_{n\to\infty}\left(1+\frac{1}{\frac{n}{3}}\right)^{\frac{n}{3}}\right]^{3} =$

$= e^{3}$

- $\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^{2n+3} =$

$= \lim\limits_{n\to\infty}\underbrace{\left(1+\frac{2}{n}\right)^{2n}}\cdot\underbrace{\left(1+\frac{2}{n}\right)^{3}}_{\to 1} = e^{4}$

$\underbrace{\left[\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right]^{4}}_{\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\to e}$

$$\downarrow$$
$$e^4$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n\,;$$

če $(a_n)_n$ in $(b_n)_n$ konvergirata proti realnemu številu

$$\bullet \lim_{n\to\infty}\left[\underbrace{\log_2(2n-1)}_{\infty} - \underbrace{\log_2(n)}_{\infty} - 1\right] =$$

$$= \lim_{n\to\infty}\left[\underbrace{\log_2\Big(\underbrace{\frac{2n-1}{n}}_{\to 2}\Big)}_{\to 1} - 1\right] = 0$$

Rekurzivno podana zaporedja

$a_0 = a$

$a_{n+1} = f(a_n)$

$a_0 = a, \quad a_1 = b$

$a_{n+2} = f(a_n, a_{n+1})$

Fibonaccijevo zaporedje

$a_0 = a_1 = 1$

$a_{n+2} = a_n + a_{n+1}$

$a_n = \sqrt{2 + \sqrt{2 + \cdots}}$, n korenov

$a_{n+1} = \sqrt{2 + \underbrace{\sqrt{2 + \cdots}}_{n \text{ korenov}}}$; $n+1$ korenov

$a_{n+1} = \sqrt{2 + a_n}$, $a_1 = \sqrt{2}$

. Dokaži, da zaporedje $(a_n)_n$ konvergira!

Poišči limito zaporedja.

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \dots$$

Naše zaporedje je naraščajoče?

Naše zaporedje je navzgor omejeno z 2?

$a_n \leq 2$ za vse $n \in \mathbb{N}$, $n \geq 1$

Indukcija po n:

$n = 1$:

$a_1 = \sqrt{2} \leq 2$ ✓

I.P. $a_n \leq 2$. Hočemo: $a_{n+1} \leq 2$

$n \Rightarrow n+1$

$$a_{n+1} = \sqrt{a_n + 2} \leq \sqrt{2+2} = 2$$ ✓

$a_n \leq 2$ po I.P.

$a_{n+1} \geq a_n$ za $n \geq 1$

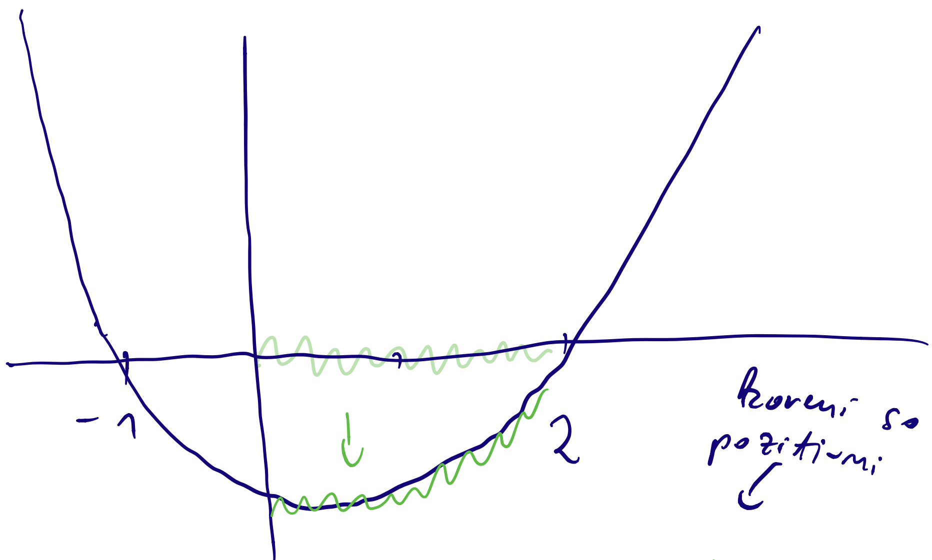
$$a_{n+1} = \sqrt{2 + a_n} \geq a_n$$

$a_n \geq 0$

$$2 + a_n \geq a_n^2$$

$$0 \geq a_n^2 - a_n - 2 =$$

$$= (a_n - 2)(a_n + 1)$$



vemo : $\forall n \geq 1$ je $0 \leq a_n \leq 2$

↳ zgoraj

Zaporedje je naraščajoče in navzgor omejeno, torej je konvergentno.

Kaj je limita?

$$a_{n+1} = \sqrt{2 + a_n}$$

$$\Big\downarrow_{n \to \infty} \qquad\qquad \Big\downarrow_{n \to \infty}$$

$$\downarrow \qquad\qquad \downarrow$$
$$a \qquad\qquad a$$

$$a = \sqrt{2+a}$$ → rešitve so kandidati za limito

$$a^2 = 2 + a$$
$$0 = (a-2)(a+1)$$

kand.: $a_1 = 2$

$\cancel{a_2 = -1}$ vsi členi zaporedja so pozitivni

$$\Rightarrow \lim_{n\to\infty} a_n = 2$$

Zaporedje, podano v rekurzivni obliki.

$$a_0 = 3$$

$$\underset{\downarrow a}{a_{n+1}} = 1 + \sqrt[3]{\underset{\downarrow a}{a_n} - 1}$$

Dokaži, da je zaporedje konvergentno in izračunaj limito.

Kandidati za limito:

$$a = 1 + \sqrt[3]{a-1}$$

$$a - 1 = \sqrt[3]{a-1} \quad / ()^3$$

$$(a-1)^3 = (a-1)$$

$$(a-1)^3 - (a-1) = 0$$

$$(a-1)\left[(a-1)^2 - 1\right] = 0$$

$$(a-1)\left[a^2 - 2a\right] = 0$$

$$a(a-1)(a-2)$$

Dobimo kandidate: $a^0 = 0$

$a^1 = 1$

$a^2 = 2$

Pretvorba rek. zaporedja v eksplicitno

podano zaporedje : $a_n = \_\_\_\_$ $a_{n+1} = 1 + \sqrt[3]{a_n - 1}$

$$3,\ 1 + \sqrt[3]{\underset{\overset{\parallel}{2^{\frac{1}{3}}}}{2}},\ 1 + \sqrt[3]{\sqrt[3]{2}} = ,\ 1 + 2^{\left(\frac{1}{3}\right)^2}$$

-1

$1+\sqrt[3]{\sqrt[3]{\sqrt[3]{2}}}$, ... ①

$\hookrightarrow 2^{\frac{1}{3^3}}$

$a_n = \left\{ 1 + \left\{ 2^{\left(\frac{1}{3}\right)^n} \right\} \to 0 \right\} \to 1$

Sklepamo: $\lim\limits_{n\to\infty} a_n = 2$

Drug način: - zaporedje je padajoče
- zaporedje je navzdol omejeno (npr. z nič) ② D.N.

$\Rightarrow$ sigurno ima le eno stekališče

①

Dokaz: $a_n = 1 + 2^{\left(\frac{1}{3}\right)^n}$ za vse $n \geq 0$

$a_0 = 3 = 1 + 2 = 1 + 2^1$ $\hookrightarrow \left(\frac{1}{3}\right)^0$

Indukcijski korak:

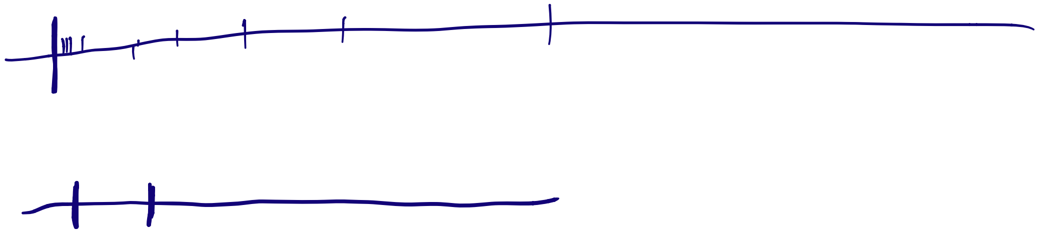
$a_n = 1 + 2^{\left(\frac{1}{3}\right)^n}$ (I.P.)

$a_{n+1} = 1 + \sqrt[3]{a_n - 1} =$

I.D.

$$= 1 + \sqrt[3]{2^{\left(\frac{1}{3}\right)^n}} = 1 + 2^{\left(\frac{1}{3}\right)^{n+1}}$$

$$\xrightarrow{n \to \infty} 1$$



$\to$ 2 stekališči

Kandidati: 0, 1, 2

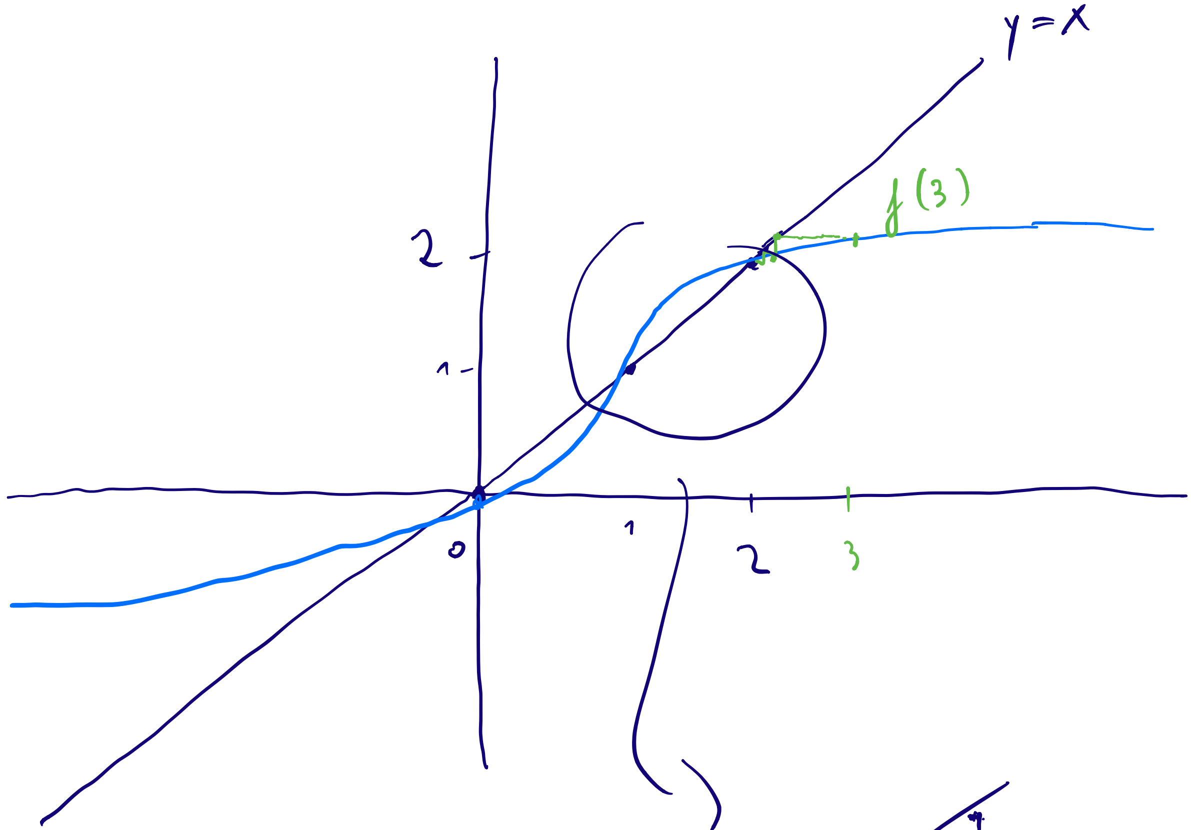
$a_0 = 3$

$a_{n+1} = 1 + \sqrt[3]{a_n - 1}$

$a_0 = 0 \Rightarrow \lim\limits_{n \to \infty} a_n = 0$

$a_1 = 1 \Rightarrow \lim\limits_{n \to \infty} a_n = 1$

$\hookrightarrow f(x) = 1 + \sqrt[3]{x-1}$; $a_{n+1} := f(a_n)$



Številske vrste

$(a_n)_n = \{a_0, a_1, a_2, \ldots\}$

$S_0 := a_0$, $S_1 := a_0 + a_1$, ..., $S_n = a_0 + \ldots + a_n$

ALI OBSTAJA

$$\lim_{n\to\infty} S_n := \sum_{n=0}^{\infty} a_n \ ?$$

Obravnaj konvergenco št. vrst!

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

PRVA IDEJA: KORENSKI KRITERIJ

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ divergira

$$R := \lim_{n\to\infty} \sqrt[n]{\frac{1}{\ln n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\ln n}} = 1$$

$\sqrt[n]{n} \longrightarrow 1$

NAMIG: za vsak $\varepsilon > 0$:

$n < (1+\varepsilon)^n$ za dovolj velik n

↳ razst. po binomski formuli

$\sqrt[n]{n} \longrightarrow 1$

$\sqrt[n]{\ln n} \longrightarrow 1$

$1 \leq \ln n \leq n$ *

↳ $n \geq 3$

$\sqrt[n]{\ }$

$1 \leq \sqrt[n]{\ln n} \leq \sqrt[n]{n}$

$\downarrow \qquad \downarrow$

$1 \qquad 1$

☆ $\Rightarrow \frac{1}{n} \leq \frac{1}{\ln n}$

$$\sum_{n=2}^{\infty} \frac{1}{n} \text{ divergira } \Rightarrow \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

$$\Bigg(\underbrace{\sum_{n=2}^{N} \frac{1}{n}}_{\substack{N \to \infty \\ \downarrow \\ \infty}} \leq \sum_{n=2}^{N} \frac{1}{\ln n} \Bigg)$$

Po primerjavi z "minoranto" sklepamo, da vrsta divergira.

- $$\sum_{n=1}^{\infty} \frac{2n-1}{2^{n-1}}$$

Ideja: KORENSKI KRITERIJ

- $$\sqrt[n]{\frac{2n-1}{2^{n-1}}} = \frac{\sqrt[n]{2n-1}}{\sqrt[n]{2^{n-1}}} \xrightarrow{n \to \infty} \frac{1}{2} < 1$$

$1 \leq \sqrt[n]{2n-1} \leq \sqrt[n]{2n} \to 1$

$\sqrt[n]{2^{n-1}} = 2^{\frac{n-1}{n}} \to 2, \quad \frac{n-1}{n} \to 1$

Po korenskem kriteriju vrsta konvergira.

- $$\sum_{n=0}^{\infty} \sqrt{\frac{n}{n^4+1}}$$

$$\sqrt{\frac{n}{n^4+1}} \leq \sqrt{\frac{n}{n^4}} = \left(\frac{1}{n}\right)^{\frac{3}{2}}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{3}{2}} < \infty$$

$$\left(\frac{1}{n}\right)^{\frac{3}{2}} \geq \sqrt{\frac{n}{n^4+1}} \geq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \sqrt{\frac{n}{n^4+1}} < \infty$$