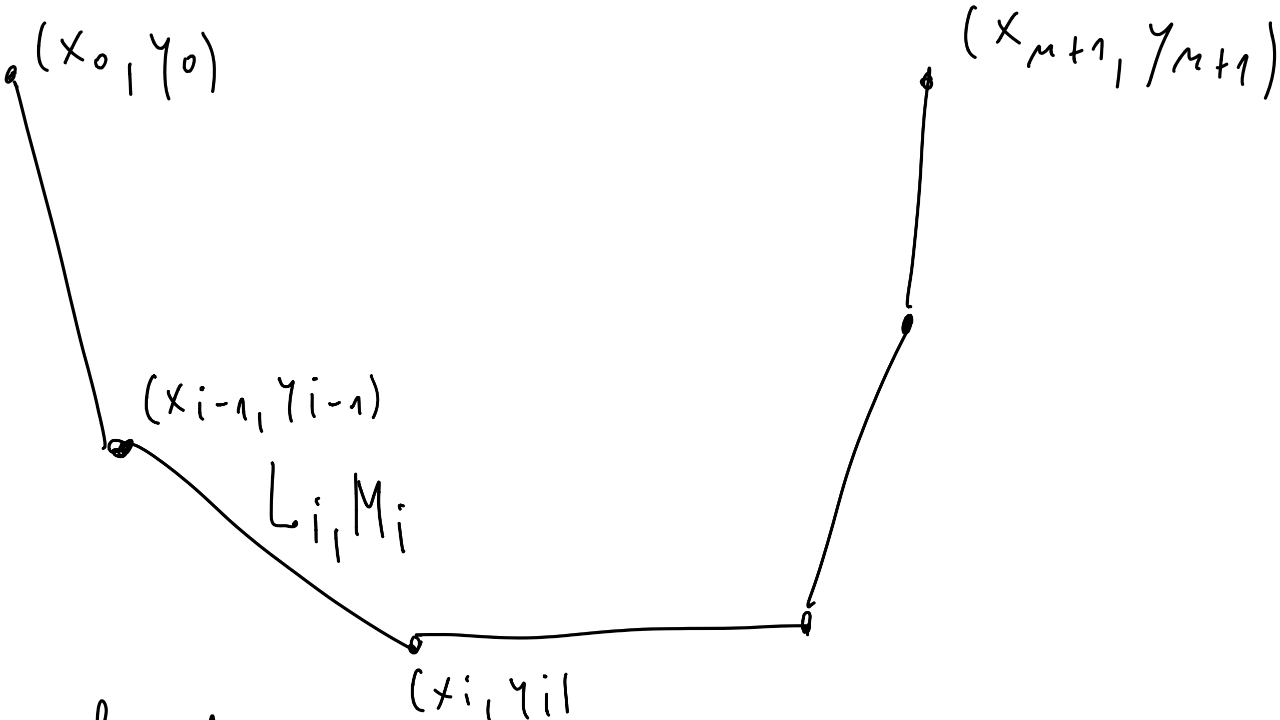
Diskretna verižnica



Diskretna verižnica je sestavljena iz $n+1$ členkov (homogenih):

L_i, $\quad i = 1, 2, \dots, n+1$; dolžina

M_i, $\quad i = 1, 2, \dots, n+1$; masa

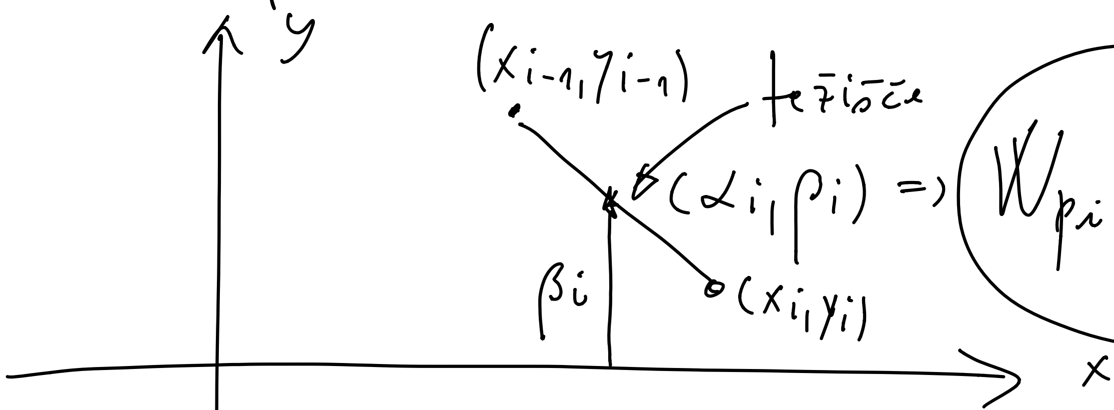
Verižnico vpnemo tako, da fiksiramo prvi členek (eno od krajišč) v (x_0, y_0), zadnjega pa v (x_{n+1}, y_{n+1}).

Če krajišča verižnice, ki določajo i-ti členek, označimo z (x_{i-1}, y_{i-1}) ter (x_i, y_i), potem bo naloga rešena, če izračunamo koordinate (x_i, y_i); $i = 1, 2, \dots, n$.

Veričnica „želi minimizirati" potencialno energijo sistema členkov!

W_p – potencialna energija sistema:

<u>Ena palica</u>:



$$\Rightarrow W_{pi} = M_i g \beta_i$$

$$\Rightarrow \beta_i = \frac{y_{i-1} + y_i}{2} \Rightarrow W_{pi} = M_i g \frac{y_{i-1} + y_i}{2}$$

Več palic: $W_p = \sum_{i=1}^{n+1} W_{pi}$

$$= \sum_{i=1}^{n+1} M_i g \frac{y_{i-1} + y_i}{2}$$

Iščemo min W_p, katerega lokacija

ni odvisna od y!!

Normirana potencialna energija:

$$\min_{x,y} F(x,y) = \min_{x,y} \sum_{i=1}^{m+1} M_i \frac{y_{i-1}+y_i}{2}$$

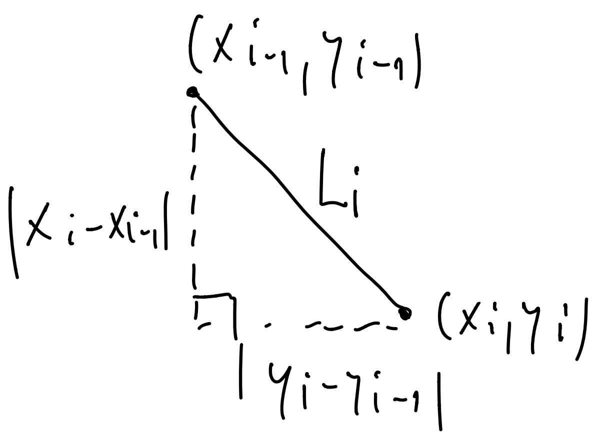
$$x = (x_0, x_1, \dots, x_{m+1})^T, \quad y = (y_0, y_1, \dots, y_{m+1})^T$$

— znamo

— neznanke

Minimum iščemo pri določenih pogojih:

$$(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 = L_i^2 \, ; \; i = 1, 2, \dots, m+1$$



Iščemo torej vezan ekstrem!

$$G(x,y,\lambda) = F(x,y) + \sum_{j=1}^{m+1} \lambda_j \big((x_j - x_{j-1})^2 +$$

$$(y_j - y_{j-1})^2 - L_j^2 \Big)$$

Iščemo klasični ekstrem (minimum) funkcije $G(x, y, \lambda)$.

Poiskati moramo stacionarne točke G:

$$\frac{1}{2}\frac{\partial G}{\partial x_i} = 0, \quad i = 1, 2, \dots, n,$$

$$\frac{1}{2}\frac{\partial G}{\partial y_i} = 0, \quad i = 1, 2, \dots, n,$$

$$\frac{\partial G}{\partial \lambda_i} = 0, \quad i = 1, 2, \dots, n.$$

$\Rightarrow$

$$\lambda_i (x_i - x_{i-1}) - \lambda_{i+1}(x_{i+1} - x_i) = 0, \quad i = 1, 2, \dots, n$$

$$\lambda_i (y_i - y_{i-1}) - \lambda_{i+1}(y_{i+1} - y_i) = -\frac{M_i + M_{i+1}}{4}, \quad i = 1, 2, \dots, n$$

$$(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 = L_i^2, \quad i = 1, 2, \dots, n+1$$

Dobimo sistem $3n+1$ nelinearnih enačb

za $3n+1$ neznank: $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, \lambda_1, \lambda_2, \ldots, \lambda_{n+1}$.

Najprej uvedemo t. i. relativne koordinate:

$$\xi_i = x_i - x_{i-1}; \quad i = 1, 2, \ldots, n+1,$$

$$\eta_i = y_i - y_{i-1}; \quad i = 1, 2, \ldots, n+1.$$

Sledi:

$$x_i = x_0 + \sum_{j=1}^{i} \xi_j \; ; \; i = 1, 2, \ldots, n+1,$$

$$y_i = y_0 + \sum_{j=1}^{i} \eta_j ; \; i = 1, 2, \ldots, n+1.$$

Sistem se prepiše v obliko

$$\lambda_i \xi_i - \lambda_{i+1} \xi_{i+1} = 0, \quad i = 1, 2, \ldots, n,$$

$$\lambda_i \eta_i - \lambda_{i+1} \eta_{i+1} = -\frac{1}{2} \mu_i, \quad i = 1, 2, \ldots, n,$$

$$\xi_i^2 + \eta_i^2 = L_i^2 ; \quad i = 1, 2, \ldots, n+1;$$

kjer je $\mu = \dfrac{M_i + M_{i+1}}{}$

2

Opazimo, da je $\lambda_i \xi_i$ konstantno za $\forall i$:

$$\lambda_i \xi_i = -\frac{1}{2u}; \quad i = 1, 2, \dots, n+1$$

neznanka

Torej je $\lambda_i = -\dfrac{1}{2u\xi_i}$, od koder dobimo

$$-\frac{1}{2u\xi_i}\eta_i + \frac{1}{2u\xi_{i+1}}\eta_{i+1} = -\frac{1}{2}\mu_i, \quad i = 1, 2, \dots, n$$

Sledi: $$\frac{\eta_{i+1}}{\xi_{i+1}} = \frac{\eta_i}{\xi_i} - u\mu_i; \quad i = 1, 2, \dots, n$$

Če nadaljujemo, dobimo

$$\frac{\eta_i}{\xi_i} = \overbrace{\frac{\eta_1}{\xi_1}}^{=:v} - u\mu_1 - u\mu_2 - \dots - u\mu_{i-1}$$

$i-1$

$$= v - u \sum_{j=1} \mu_j \; ; \; i = 1, 2, \dots, n+1.$$

Upoštevamo

$$\xi_i^2 + \eta_i^2 = L_i^2 \quad / : \xi_i^2$$

$$1 + \left(\frac{\eta_i}{\xi_i}\right)^2 = \frac{L_i^2}{\xi_i^2}$$

Od tod dobimo

$$\xi_i = \pm \frac{L_i}{\sqrt{1 + \left(v - u \sum_{j=1}^{i-1} \mu_j\right)^2}} \; ; \; i = 1, 2, \dots, n+1$$

Nadalje izrazimo

$$\eta_i = \xi_i \left(v - u \sum_{j=1}^{i-1} \mu_j\right)$$

Dobili smo, da je $\xi_i = \xi_i(u, v)$

$\eta_i = \eta_i(u, v)$

$$\sum^{n+1} \Big(\dots \Big)$$

$$U(u,v) = \sum_{i=1} \left(\xi_i - (x_{m+1} - x_0) \right)$$

$$= \cancel{x_1} - \cancel{x_0} + \cancel{x_2} - \cancel{x_1} + \cancel{x_3} - \cancel{x_2} + \dots + \cancel{x_{m+1}} - \cancel{x_m}$$

$$- \cancel{x_{m+1}} + \cancel{x_0} = 0 \Rightarrow \boxed{U(u,v) = 0}$$

$$V(u,v) = \sum_{i=1}^{m+1} \left(\eta_i - (y_{m+1} - y_0) \right) = 0 \Rightarrow \boxed{V(u,v) = 0}$$

Kako rešujemo sistem

$$U(u,v) = 0$$
$$V(u,v) = 0 \quad ?$$

- Lahko uporabimo Newtonovo metodo v dveh dimenzijah:

$$H(u,v) = \begin{bmatrix} U(u,v) \\ V(u,v) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$(u,v) = X$:

$$H(X) = \begin{bmatrix} U(X) \\ V(X) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

$$X^{(r+1)} = X^{(r)} - J^{-1}(X^{(r)})\, H(X^{(r)})$$

$$X = X - J_H(X) \cdot H(X), \; r = 0, 1, \dots$$

Namesto tega rešujemo sistem:

$$\underbrace{J_H(X^{(r)})\, \Delta X^{(r)} = -H(X^{(r)})}_{\boxed{A \cdot x = b}}$$

- Lahko pa uporabite $\boxed{\text{fsolve}}$ za reševanje sistema nel. enačb (optimization toolbox).

- Lahko tudi uporabite fmincon na začetnem problemu!