LINEARNA ALGEBRA 2020/21
6. VAJE: 9. 11. 2020

1. Parametrično izrazi rešitev sistema enačb.

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 2 & 0 \\ -2 & 2 & -1 & -4 & -3 & 0 \\ 1 & -1 & 1 & 3 & 1 & 0 \\ -1 & 1 & 1 & 1 & -3 & 0 \end{array}\right],$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 3 & 3 & 4 & 2 \\ -2 & 2 & -1 & -3 & -4 & -5 & -1 \\ -1 & 1 & 0 & 0 & -1 & -4 & -5 \\ -1 & 1 & 1 & 3 & 1 & -1 & -2 \end{array}\right].$$

2. Določi za katere vrednosti a je sistem rešljiv in ga za dobljene vrednosti reši.

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 0 & a+1 \\ -1 & 2 & 3 & 1 & -2a \\ 1 & -1 & -1 & 1 & 0 \\ 1 & 1 & 3 & 1 & a-1 \end{array}\right]$$

3. Podane so matrike

$$A = \begin{bmatrix} 2 & 3 & -2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix} \text{ in } D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

   Izračunaj vse možne produkte oblike XY za $X, Y \in \{A, B, C, D\}$.

4. Naj bo I identična matrika ($n \times n$ matrika, ki ima na diagonali 1 izven pa 0). Pokaži, da je za katerokoli matriko primerne velikosti veja $AI = A$ ali $IA = A$.

5. Izračunaj

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^n, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^n, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n \text{ in } \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^n, \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}^n.$$

6. Pokaži, da je $(AB)C = A(BC)$ za poljubne matrike A, B in C, ki jih lahko zmnožimo na zapisan način.

homogen

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 2 & 0 \\ -2 & 2 & -1 & -4 & -3 & 0 \\ 1 & -1 & 1 & 3 & 1 & 0 \\ -1 & 1 & 1 & 1 & -3 & 0 \end{array}\right] \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix}$$

$$x_1 - x_2 + x_4 + 2x_5 = 0$$
$$x_3 + 2x_4 - x_5 = 0$$

$x_2 = t_1$ $\qquad x_1 = t_1 - t_2 - 2t_3$

$x_4 = t_2$ $\qquad x_3 = -2t_2 + t_3$

$x_5 = t_3$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

nehomogen

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 3 & 3 & 4 & 2 \\ -2 & 2 & -1 & -3 & -4 & -5 & -1 \\ -1 & 1 & 0 & 0 & -1 & -4 & -5 \\ -1 & 1 & 1 & 3 & 1 & -1 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccccc|c} 1 & -1 & 1 & 3 & 3 & 4 & 2 \\ 0 & 0 & 1 & 3 & 2 & 3 & 3 \\ 0 & 0 & 1 & 3 & 2 & 0 & -3 \\ 0 & 0 & 2 & 6 & 4 & 3 & 0 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 3 & 3 & 4 & 2 \\ 0 & 0 & 1 & 3 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 & -3 & -6 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccccc|c} 1 & -1 & 1 & 3 & 3 & 4 & 2 \\ 0 & 0 & 1 & 3 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccccc|c} 1 & -1 & 1 & 3 & 3 & 0 & -6 \\ & & 1 & 3 & 2 & 0 & -3 \\ & & & & & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 1 & 0 & -3 \\ & & 1 & 3 & 2 & 0 & -3 \\ & & & & & 1 & 2 \end{array}\right]$$

$$x_1 - x_2 + x_5 = -3$$
$$x_3 + 3x_4 + 2x_5 = -3$$
$$x_6 = 2$$

$x_2 = t_1$

$x_4 = t_2$

$x_5 = t_3$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -3 \\ 0 \\ 0 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 0 & a+1 \\ -1 & 2 & 3 & 1 & -2a \\ 1 & -1 & -1 & 1 & 0 \\ 1 & 1 & 3 & 1 & a-1 \end{array}\right].$$

Kdaj je sistem rešljiv?
Kaj je rešitev?

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & a+1 \\ -1 & 2 & 3 & 1 & -2a \\ 1 & 1 & 3 & 1 & a-1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & a+1 \\ 0 & 1 & 2 & 2 & -2a \\ 0 & 2 & 4 & 0 & a-1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & \frac{a-1}{2} \\ 0 & 1 & 2 & -1 & \frac{a+1}{2} \\ 0 & 1 & 2 & 2 & -2a \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & \frac{a-1}{2} \\ 0 & 0 & 0 & -1 & 1 = \frac{a+1}{2} - \frac{a-1}{2} \\ 0 & 0 & 0 & 2 & \frac{-4a-a+1}{2} \end{array}\right] \sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ & 1 & 2 & & \frac{a-1}{2} \\ & & & 1 & -1 \\ & & & 1 & \frac{-5a+1}{4} \end{array}\right).$$

$x_4 = -1$ $\qquad x_4 = x_4$

$x_4 = \frac{-5a+1}{4}$ $\qquad -1 = \frac{-5a+1}{4}$ $\qquad -5a+1 = -4$ $\qquad -5a = -5$ $\qquad \underline{a = 1}$

Če $a \neq 1$, sistem ni rešljiv. (Potem je $-1 = x_4 = \frac{-5a+1}{4} \neq -1$)

$a=1$

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ & 1 & 2 & & 0 \\ & & & 1 & -1 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & \\ & & & 1 & -1 \end{array}\right) \overset{+1}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ & 1 & 2 & 0 & 0 \\ & & & 1 & -1 \end{array}\right)$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

$x_1 + x_3 = 1$ $\qquad x_1 = 1 - x_3$

$x_2 + 2x_3 = 0$ $\qquad x_2 = -2x_3$

$x_4 = -1$ $\qquad x_4 = -1$

$x_3 = t_1$

$m \times n$ matrika

$A = (a_{ij})_{\substack{i=1\dots m \\ j=1\dots n}}$ $\qquad$ $B = (b_{ij})_{\substack{i=1\dots m' \\ j=1\dots n'}}$

$n = m'$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix} m$$

$A \cdot B$

$a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \cdots + a_{1n}b_{n1}$

$$m \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n'} \\ b_{21} & b_{22} & & b_{2n'} \\ \vdots & & & \\ b_{n1} & & & b_{nn'} \end{bmatrix} n \qquad \begin{bmatrix} 0 & 0 & & \\ & & 0_{ij} & \\ \cdots & & & \end{bmatrix} m$$

skalarni produkt i-te vrstice in j-tega stolpca

$AB = C \qquad C = (c_{ij})$

$(a_{ij})_{\substack{i=1\dots m \\ j=1\dots n}} \quad (b_{ij})_{\substack{i=1\dots n \\ j=1\dots n'}}$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

skal. prod.

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

3. Podane so matrike

$$A = \begin{bmatrix} 2 & 3 & -2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix} \text{ in } D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Izračunaj vse možne produkte oblike XY za $X, Y \in \{A, B, C, D\}$.

~~AA~~	BA ✓	~~CA~~	~~DA~~
AB ✓	~~BB~~	~~CB~~	~~DB~~
AC ✓	~~BC~~	~~CC~~	~~DC~~
~~AD~~	~~BD~~	CD ✓	DD ✓

$$A \cdot B = \begin{bmatrix} 2 & 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 3\cdot 3 - 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}$$

1×1 matrika

$$A \cdot C = \begin{bmatrix} 2 & 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 - 3\cdot 1 - 8, & 2\cdot 1 - 1 \end{bmatrix} = \begin{bmatrix} -5 & 1 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -2 & -1 \\ 6 & 9 & -6 & -3 \\ 4 & 6 & -4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C \cdot D = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & -1 \\ 4 & 4 \\ 0 & 1 \end{bmatrix}$$

$$D \cdot D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(1) $A \cdot B \overset{\text{v splošnem}}{\neq} B \cdot A$

(2) $A \cdot B = A \cdot C \Rightarrow$ ni nujno $B = C$

$(A+B)C = AC + BC$ $\qquad$ $A(B+C) = AB + AC$

4. Naj bo I identična matrika ($n \times n$ matrika, ki ima na diagonali 1 izven pa 0). Pokaži, da je za katerokoli matriko primerne velikosti veja $AI = A$ ali $IA = A$.

$$\underbrace{I_n}_{\text{enota za množenje}} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$AI = A$

$IA = A$ (če lahko množimo)

npr.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$I = (\delta_{ij})_{i,j=1}^{n} \qquad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{sicer} \end{cases}$$ Kroneckerjeva delta

$A = (a_{ij})_{\substack{j=1 \ldots n \\ i=1 \ldots m}} {}_{\substack{j=1 \ldots n \\ i=1 \ldots n}}$

AI izračunamo ij-ti koeficient

$$\sum_{k=1}^{n} a_{ik} \underbrace{\delta_{kj}}_{\text{če je } k=j, \text{ dobimo nekaj (potencialno) neničelnega}} = a_{ij}$$

IA $\qquad \sum_{k=1}^{n} \underbrace{\delta_{ik}}_{i=k} a_{kj} = a_{ij}$

samo ko je $k=i$, dobimo nekaj.

5. Izračunaj

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}^n, \begin{bmatrix}1&1\\0&1\end{bmatrix}^n, \begin{bmatrix}0&1\\1&0\end{bmatrix}^n, \begin{bmatrix}0&-1\\1&0\end{bmatrix}^n \text{ in } \begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}^n, \begin{bmatrix}0&a&c\\0&0&b\\0&0&0\end{bmatrix}^n.$$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}^n = \underline{I}\begin{bmatrix}1&\\&1\end{bmatrix}^{n-1} = \begin{bmatrix}1&\\&1\end{bmatrix}^{n-1} \cdots = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$A = \begin{bmatrix}1&1\\0&1\end{bmatrix}$$

$$A^n = \begin{bmatrix}1&n\\0&1\end{bmatrix}$$

Z indukcijo

$n=1 \qquad A^1 = A = \begin{bmatrix}1&1\\0&1\end{bmatrix}$

Recimo, da velja za n

$$A^{n+1} = A A^n = \begin{bmatrix}1&1\\&1\end{bmatrix}\begin{bmatrix}1&n\\&1\end{bmatrix} = \begin{bmatrix}1&n+1\\0&1\end{bmatrix} \checkmark$$

$$B = \begin{bmatrix}0&1\\1&0\end{bmatrix} \qquad B^2 = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I_2$$

$$B^{2n} = B^{2n} = (B^2)^n = (I_2)^n = I_2 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$B^{2n+1} = B \cdot \underline{B^{2n}} = B \cdot I = B$$

$$C = \begin{bmatrix}0&-1\\1&0\end{bmatrix} \qquad C^n = \qquad C^2 = \begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}-1&0\\0&-1\end{bmatrix} = -I$$

$$C^3 = C^2 \cdot C = -IC = -C$$

$$C^4 = C^2 \cdot C^2 = (-I)^2 = I$$

$$C^5 = I$$

$$C^{4n} = I$$

$$C^{4n+1} = \underline{C^{4n}} \cdot C = C$$

$$C^{4n+2} = C^{4n} \cdot C^2 = -I$$

$$C^{4n+3} = C^{4n} C^3 = -C$$

$$a_1 + ib_1 \in \mathbb{C} \rightsquigarrow a_1 I + b_1 C = \begin{bmatrix}a_1&-b_1\\b_1&a_1\end{bmatrix}$$

$$a_2 + ib_2 \in \mathbb{C} \rightsquigarrow a_2 I + b_2 C = \begin{bmatrix}a_2&-b_2\\b_2&a_2\end{bmatrix}$$

$$(a_1+ib_1)(a_2+ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$

$$\begin{bmatrix}a_1&-b_1\\b_1&a_1\end{bmatrix} \cdot \begin{bmatrix}a_2&-b_2\\b_2&a_2\end{bmatrix} = \begin{bmatrix}a_1a_2-b_1b_2 & -a_1b_2-a_2b_1\\ a_1b_2+a_2b_1 & a_1a_2-b_1b_2\end{bmatrix}$$

$$D = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \qquad D^n = \begin{bmatrix} a^n & & \\ & b^n & \\ & & c^n \end{bmatrix}$$

Indukcija

$n=1$ ✓

$n \longrightarrow n+1$

$$D \cdot D^n = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \begin{bmatrix} a^n & & \\ & b^n & \\ & & c^n \end{bmatrix} = \begin{bmatrix} a^{n+1} & 0 & 0 \\ 0 & b^{n+1} & 0 \\ 0 & 0 & c^{n+1} \end{bmatrix} \checkmark$$

Diagonalna $m \times m$ matrika

$$D = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_m \end{bmatrix} \qquad D^n = \begin{bmatrix} a_1^n & & & \\ & a_2^n & & \\ & & \ddots & \\ & & & a_m^n \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \quad E^2 = \begin{bmatrix} 0 & 0 & a \cdot c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E^3 = \begin{bmatrix} & & a \cdot c \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & a & b \\ & & c \\ & & \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$n \geq 3$

$$E^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$m \times m$ - matrika

$$E' = \begin{bmatrix} 0 & & & * \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix} \qquad E'^{\,m} = 0. \quad \left(E'^{\,m'} = 0 \right)$$

za vsak $m' \geq m$.

strogo zgornje trikotna matrika

6.

$$(AB)C = A(BC)$$ množenje matrik je asociativno.

$A = (a_{ij})$

$B = (b_{ij})$

$C = (c_{ij})$

ij-ti koeficient matrike AB

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\big((AB)C\big)_{ij} = \sum_{l=1}^{n'} \left(\sum_{k=1}^{n} a_{ik} b_{kl} \right) c_{lj} = \sum_{l=1}^{n'} \sum_{k=1}^{n} a_{ik} b_{kl} c_{lj}$$

$$(BC)_{ij} = \sum_{k=1}^{n'} b_{ik} c_{kj}$$

Vsota je ista. (Samo indeksi imajo drugačno oznako) $\sum_{j=1}^{n} x_j = \sum_{i=1}^{n} x_i$

$$\big(A(BC)\big)_{ij} = \sum_{l=1}^{n} a_{il} \left(\sum_{k=1}^{n'} b_{lk} c_{kj} \right) = \sum_{l=1}^{n} \sum_{k=1}^{n'} a_{il} b_{lk} c_{kj}$$