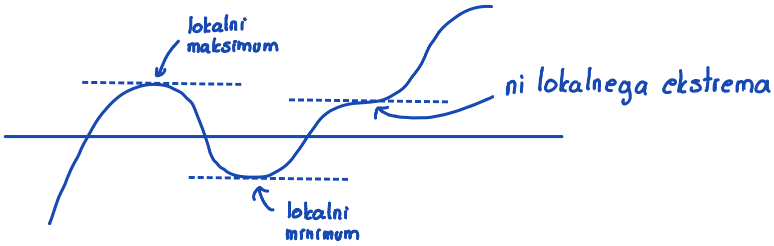
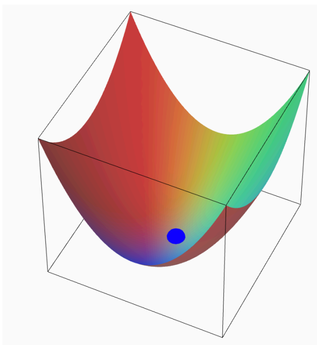
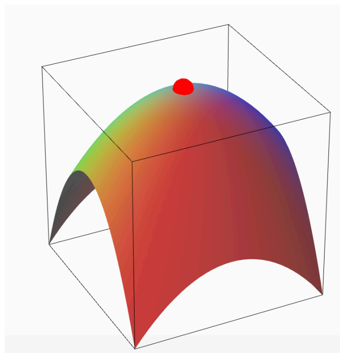
LOKALNI EKSTREMI

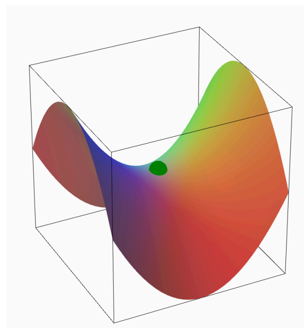
funkcije ene spremenljivke:



Funkcije dveh spremenljivk:


lokalni minimum


lokalni maksimum


sedlo

Lokalni ekstremi funkcij dveh spremenljivk

Točka (a, b) funkcije f dveh spremenljivk je **stacionarna**, če je $f_x(a, b) = f_y(a, b) = 0$.

Če ima funkcija f dveh spremenljivk v notranji točki (a, b) lokalni ekstrem, je stacionarna.

Hessejeva matrika:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$K(a, b) = det H(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - f_{xy}(a, b) \cdot f_{yx}(a, b)$$

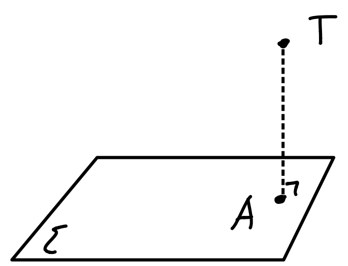
Naj bo f dvakrat parcialno zvezno odvedljiva in naj bo v (a, b) stacionarna točka.

- $K(a, b) > 0$ in $f_{xx}(a, b) > 0$: v (a, b) lokalni minimum
- $K(a, b) > 0$ in $f_{xx}(a, b) < 0$: v (a, b) lokalni maksimum
- $K(a, b) < 0$: v (a, b) ni lokalnega ekstrema (sedlo)
- $K(a, b) = 0$: v (a, b) ne vemo, ali je ekstrem

31. Na ravnini z enačbo $x - 2y + z = 4$ poišči točko, ki je najbližja točki $(-1, 0, -1)$.

$\Sigma : x - 2y + z = 4$

$T(-1, 0, -1)$



$A \ldots$ točka na Σ $\quad A(x, y, 4 - x + 2y)$

$$d(A, T) = \sqrt{(x+1)^2 + y^2 + (5 - x + 2y)^2}$$

Iščemo minimum funkcije $d(A(x, y, 4 - x + 2y), T)$.

Dosežen je v isti točki (x, y) kot minimum funkcije

$$f(x, y) = \left(d(A(x, y, 4 - x + 2y), T)\right)^2 = (x+1)^2 + y^2 + (5 - x + 2y)^2$$

$$f_x(x, y) = 2 \cdot (x+1) + 2 \cdot (5 - x + 2y) \cdot (-1) = 2x + 2 - 10 + 2x - 4y = 4x - 4y - 8$$

$$f_y(x, y) = 2 \cdot y + 2 \cdot (5 - x + 2y) \cdot 2 = 2y + 20 - 4x + 8y = -4x + 10y + 20$$

Kandidati za lokalne ekstreme (stacionarne točke):

$f_x(x, y) = 0 \qquad\qquad f_y(x, y) = 0$

$4x - 4y - 8 = 0 \;/ :4 \qquad -4x + 10y + 20 = 0 \;/ :2$

$x - y - 2 = 0 \qquad\qquad -2x + 5y + 10 = 0$

$y = x - 2 \qquad \longrightarrow \qquad -2x + 5 \cdot (x - 2) + 10 = 0$

$$-2x + 5x \cancel{-10} + \cancel{10} = 0$$

$$x = 0 \qquad \Rightarrow \quad y = 0 - 2 = -2$$

stacionarna točka: $T(0, -2)$

Ali je stacionarna točka $T(0,0)$ lokalni ekstrem?

$f_{xx}(x,y) = 4$

$f_{xy}(x,y) = -4$

$f_{yy}(x,y) = 10$

$$H(0,-2) = \begin{array}{c} \begin{array}{cc} 1^{\circ} & \end{array} \\ \begin{bmatrix} 4 & -4 \\ -4 & 10 \end{bmatrix} \end{array} \qquad \det H(0,-2) = 24 > 0 \qquad \Rightarrow T(0,-2) \text{ je lok. minimum}$$

$$f_{xx}(0,-2) > 0$$

$f(x,y) \to \infty$, ko $x \to \infty$ ali $y \to \infty$ $\Rightarrow$ $(0,-2)$ je globalni minimum.

$\Rightarrow$ Najbližja točka je $A(0,-2,0)$.

32. Določite stacionarne točke in lokalne ekstreme funkcij f, podanih z naslednjimi predpisi:

a) $f(x,y) = x^4 + 4xy + y^4 + 1$

b) $f(x,y) = e^{-x}(x - y^2)$

c) $f(x,y,z) = (3x^2 + 2y^2 + z^2 - 2xy - 2yz)e^{-x}$

d) $f(x,y) = (x+y) \cdot e^{x-y}$

e) $f(x,y) = (x-y)\ln(x+y)$

f) $f(x,y) = \frac{x}{y} - \ln x + y^2 - y$

g) $f(x,y) = x^3 + 8y^3 - 6xy + 5$

h) $f(x,y) = (x-y)(x+y)(y-1)$

i) $f(x,y) = x^2y^2$

j) $f(x,y,z) = xyz(x+y+z-4)$ za $x,y,z > 0$

a) $f(x,y) = x^4 + 4xy + y^4 + 1$

$$\left.\begin{array}{lll} f_x(x,y) = 4x^3 + 4y = 0 & \Leftrightarrow & y = -x^3 \\ f_y(x,y) = 4x + 4y^3 = 0 & \Leftrightarrow & x = -y^3 \end{array}\right\}$$

$x = x^9$
$x^9 - x = 0$
$x \cdot (x^8 - 1) = 0$
$x \cdot (x^4 - 1) \cdot (x^4 + 1) = 0$
$x \cdot (x^2 - 1) \cdot (x^2 + 1) \cdot (x^4 + 1) = 0$
$x \cdot (x-1) \cdot (x+1) \cdot (x^2 + 1) \cdot (x^4 + 1) = 0$

$x_1 = 0 \qquad x_2 = 1 \qquad x_3 = -1$
$y_1 = 0 \qquad y_2 = -1 \qquad y_3 = 1$

Stacionarne točke: $T_1(0,0)$, $T_2(1,-1)$, $T_3(-1,1)$

$f_{xx}(x,y) = 12x^2$

$f_{xy}(x,y) = 4$

$f_{yy}(x,y) = 12y^2$

$H(0,0) = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$ $\qquad$ $\det H(0,0) = 16 < 0$ $\Rightarrow$ u $T_1(0,0)$ ni lok. ekstrema

$H(1,-1) = \begin{bmatrix} \overset{>0}{12} & 4 \\ 4 & 12 \end{bmatrix}$ $\qquad$ $\det H(1,-1) = 128 > 0$, $f_{xx}(1,-1) > 0 \Rightarrow$ u $T_2(1,-1)$ lok. min.

$H(-1,1) = \begin{bmatrix} \overset{>0}{12} & 4 \\ 4 & 12 \end{bmatrix}$ $\qquad$ $\det H(-1,1) = 128 > 0$, $f_{xx}(-1,1) > 0 \Rightarrow$ u $T_2(-1,1)$ lok. min.

b) $f(x,y) = e^{-x}(x - y^2)$

$f_x(x,y) = -e^{-x}(x - y^2) + e^{-x} \cdot 1 = \overbrace{e^{-x}}^{>0} \cdot (-x + y^2 + 1) = 0$

$f_y(x,y) = -2ye^{-x} = 0$

$\begin{array}{r} -x + y^2 + 1 = 0 \\ y = 0 \end{array} \quad \Rightarrow \quad \begin{array}{r} -x + 1 = 0 \\ x = 1 \end{array} \quad \Rightarrow T(1,0)$ stacionarna točka

$f_{xx}(x,y) = e^{-x}(-x+y^2+1) + e^{-x} \cdot (-1) = e^{-x} \cdot (x-y^2-2)$

$f_{xy}(x,y) = 2ye^{-x}$

$f_{yy}(x,y) = -2e^{-x}$

$$H(1,0) = \begin{bmatrix} \overset{<0}{-e^{-1}} & 0 \\ 0 & -2e^{-1} \end{bmatrix}$$

$\det H(1,0) = 2e^{-2} > 0$, $f_{xx}(1,0) = -e^{-1} < 0$ $\Rightarrow$ $T(1,0)$ je lok. maks.

Lokalni ekstremi funkcij več kot dveh spremenljivk

Točka a funkcije f več spremenljivk je **stacionarna**, če je $\operatorname{grad} f(a) = \vec{0}$. Če ima funkcija f več spremenljivk v notranji točki a lokalni ekstrem, je stacionarna.

Hessejeva matrika:

$$H = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{bmatrix}$$

Ali je v dani točki ekstrem, je odvisno od lastnih vrednosti Hessejeve matrike:

- če so vse lastne vrednosti strogo pozitivne, je v dani točki lokalni minimum
- če so vse lastne vrednosti strogo negativne, je v dani točki lokalni maksimum
- ekstrema ni, če je ena lastna vrednost strogo pozitivna, druga pa strogo negativna
- v ostalih primerih, ne moremo opredeliti, ali je v dani točki ekstrem ali ne

Dodatek: iskanje lastnih vrednosti matrike.

$A \in \mathbb{C}^{n\times n}$ ... kvadratna matrika velikosti $n \times n$

Kompleksno število λ je lastna vrednost matrike A, če obstaja tak neničelen vektor

$x=(x_1,\dots,x_n)\in\mathbb{R}^n$, da velja:

$$A\cdot x = \lambda\cdot x$$
$$A\cdot x = \lambda\cdot I\cdot x \qquad I=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}$$
$$A\cdot x-\lambda\cdot I\cdot x = 0$$
$$(A-\lambda\cdot I)\cdot x = 0 \qquad \dots\ (A-\lambda\cdot I) \text{ je matrika}$$
$$\Downarrow$$
$$x\in \mathrm{Ker}(A-\lambda I)$$

$\Rightarrow$ λ je lastna vrednost matrike A $\Leftrightarrow$ $\mathrm{Ker}(A-\lambda I)\neq\{0\}$ $\Leftrightarrow$

$\Leftrightarrow$ $A-\lambda I$ ni obrnljiva $\Leftrightarrow$

$\Leftrightarrow$ $\det(A-\lambda I)=0$

Polinom $p_A(\lambda)=\det(A-\lambda I)$ v spremenljivki λ imenujemo KARAKTERISTIČNI POLINOM.

IZREK: Kompleksno število λ je lastna vrednost matrike A $\Leftrightarrow$ ko je λ ničla kar. polinoma.

Več o determinantah:

https://www.fmf.uni-lj.si/~kosir/poucevanje/skripta/determinante.pdf

(razvoj determinante, računanje determinant in lastnosti determinant)

c) $f(x,y,z) = (3x^2 + 2y^2 + z^2 - 2xy - 2yz)e^{-x}$

$$f_x(x,y,z) = (6x-2y)e^{-x} - (3x^2+2y^2+z^2-2xy-2yz)e^{-x} =$$

$$= (6x-2y-3x^2-2y^2-z^2+2xy+2yz)e^{-x} = 0$$

$$\left.\begin{array}{l} f_y(x,y,z) = (4y-2x-2z)e^{-x} = 0 \Rightarrow 2y = x+z \Rightarrow y = x \\ f_z(x,y,z) = (2z-2y)e^{-x} = 0 \Rightarrow z = y \end{array}\right\} z = y = x$$

$$6x-2y-3x^2-2y^2-z^2+2xy+2yz = 0$$

$$6x-2x-3\cancel{x^2}-2x^2-\cancel{x^2}+\cancel{2x^2}+\cancel{2x^2} = 0$$

$$4x-2x^2 = 0$$

$$x\cdot(2-x) = 0$$

$$\begin{array}{ll} x_1 = 0 & x_2 = 2 \\ y_1 = 0 & y_2 = 2 \\ z_1 = 0 & z_2 = 2 \end{array} \Rightarrow$$

stacionarni točki

$T_1(0,0,0)$

in

$T_2(2,2,2)$

$$f_{xx}(x,y,z) = (6-6x+2y)e^{-x} - f_x(x,y,z)$$

$$f_{xy}(x,y,z) = (-2-4y+2x+2z)e^{-x}$$

$$f_{xz}(x,y,z) = (-2z+2y)e^{-x}$$

$$f_{yy}(x,y,z) = 4e^{-x}$$

$$f_{yz}(x,y,z) = -2e^{-x}$$

$$f_{zz}(x,y,z) = 2e^{-x}$$

$$H(0,0,0) = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

Poiščemo lastne vrednosti $H(0,0,0)$:

$$\begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 4-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{vmatrix} = (6-\lambda)\begin{vmatrix} 4-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & -2 \\ 0 & 2-\lambda \end{vmatrix} =$$

$$= (6-\lambda)\cdot(8-6\lambda+\lambda^2-4) + 2\cdot(-4+2\lambda-0) =$$

$$= (6-\lambda)(\lambda^2-6\lambda+4) - 8 + 4\lambda =$$

$$= -\lambda^3+6\lambda^2+6\lambda^2-\cancel{4\lambda}-36\lambda+24-8+\cancel{4\lambda} =$$

$$= -\lambda^3+12\lambda^2-36\lambda+16 =$$

$$= -(\lambda-4)(\lambda^2-8\lambda+4) =$$

$$= -(\lambda-4)((\lambda-4)^2-12) =$$

$$= -(\lambda-4)(\lambda-4+2\sqrt{3})(\lambda-4-2\sqrt{3})$$

$\lambda_1 = 4 > 0 \qquad \lambda_2 = 4-2\sqrt{3} > 0 \qquad \lambda_3 = 4+2\sqrt{3} > 0$

$\Rightarrow T_1(0,0,0)$ je lokalni minimum

$$H(2,2,2) = \begin{bmatrix} -2e^{-2} & -2e^{-2} & 0 \\ -2e^{-2} & 4e^{-2} & -2e^{-2} \\ 0 & -2e^{-2} & 2e^{-2} \end{bmatrix}$$

Ta matrika je simetrična in ima nekaj pozitivnih in nekaj negativnih diagonalcev.

$\Rightarrow$ Ima nekaj pozitivnih in nekaj negativnih l. vr.

$\Rightarrow T_2(2,2,2)$ ni lokalni ekstrem.

e) $f(x,y) = (x-y)\ln(x+y)$

dodatno (nismo naredili na vajah)

10.11.2020

$$\left.\begin{aligned} f_x(x,y) &= \ln(x+y) + (x-y)\cdot\frac{1}{x+y} = 0 \\ f_y(x,y) &= -\ln(x+y) + (x-y)\cdot\frac{1}{x+y} = 0 \end{aligned}\right\} + \quad \Rightarrow 2\cdot\frac{x-y}{x+y} = 0 \Rightarrow x = y$$

$$\ln(x+x) + 0 = 0$$
$$\ln(2x) = 0$$
$$2x = 1$$
$$x = \frac{1}{2} \quad y = \frac{1}{2}$$

Stacionarna točka : $T(\frac{1}{2},\frac{1}{2})$

$$f_{xx}(x,y) = \frac{1}{x+y} + \frac{(\cancel{x}+y)-(\cancel{x}-y)}{(x+y)^2} = \frac{x+3y}{(x+y)^2} \qquad f_{xx}(\tfrac{1}{2},\tfrac{1}{2}) = 2$$

$$f_{xy}(x,y) = \frac{1}{x+y} + \frac{-(x+\cancel{y})-(x-\cancel{y})}{(x+y)^2} = \frac{-x+y}{(x+y)^2} \qquad f_{xy}(\tfrac{1}{2},\tfrac{1}{2}) = 0$$

$$f_{yy}(x,y) = -\frac{1}{x+y} + \frac{-(x+\cancel{y})-(x-\cancel{y})}{(x+y)^2} = \frac{-3x-y}{(x+y)^2} \qquad f_{yy}(\tfrac{1}{2},\tfrac{1}{2}) = -2$$

$$H(\tfrac{1}{2},\tfrac{1}{2}) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\det H(\frac{1}{2},\frac{1}{2}) = -4 < 0 \Rightarrow T(\frac{1}{2},\frac{1}{2})$ ni lokalni ekstrem.

f) $f(x,y) = \frac{x}{y} - \ln x + y^2 - y$

$D_f = \{(x,y) \in \mathbb{R}^2;\ y \neq 0 \wedge x > 0\}$

$f_x(x,y) = \frac{1}{y} - \frac{1}{x} = 0 \longrightarrow x = y$

$f_y(x,y) = -\frac{x}{y^2} + 2y - 1 = 0 \longrightarrow -\frac{\cancel{x}}{x^{\cancel{2}}} + 2x - 1 = 0$

$-\frac{1}{x} + 2x - 1 = 0 \quad / \cdot x$

$-1 + 2x^2 - x = 0$

$2x^2 - x - 1 = 0$

$(2x+1)(x-1) = 0$

$x_1 = 1 \qquad y_1 = 1$

$x_2 = -\frac{1}{2} \qquad y_2 = -\frac{1}{2}$

Stacionarni točki: $T_1(1,1)$ $\cancel{T_2(-\frac{1}{2}, -\frac{1}{2})}$

ker ni u D_f

$f_{xx}(x,y) = \frac{1}{x^2}$ $\qquad f_{xx}(1,1) = 1$

$f_{xy}(x,y) = -\frac{1}{y^2}$ $\qquad f_{xy}(1,1) = -1$

$f_{yy}(x,y) = 2\frac{x}{y^3} + 2$ $\qquad f_{yy}(1,1) = 4$

$H(1,1) = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$

$\det H(1,1) = 4 - 1 = 3 > 0$, $f_{xx}(1,1) > 0$

$\Rightarrow$ $T(1,1)$ je lokalni minimum

g) $f(x,y) = x^3 + 8y^3 - 6xy + 5$ dodatno (nismo naredili na vajah)

$f_x(x,y) = 3x^2 - 6y = 0 \longrightarrow x^2 = 2y$

$f_y(x,y) = 24y^2 - 6x = 0 \longrightarrow 4y^2 = x$

$16y^4 = 2y$

$2 \cdot y \cdot (8y^3 - 1) = 0$

$y \cdot (2y-1)(4y^2+2y+1) = 0$

$y \cdot (2y-1) \cdot \left((2y+\frac{1}{2})^2 - \frac{1}{4} + 1\right) = 0$

$y \cdot (2y-1) \cdot \left((2y+\frac{1}{2})^2 + \frac{3}{4}\right) = 0$

$y \cdot (2y-1) = 0$

$y_1 = 0 \qquad y_2 = \frac{1}{2}$

$x_1 = 0 \qquad x_2 = 4 \cdot (\frac{1}{2})^2 = 1$

Stacionarni točki: $T_1(0,0)$, $T_2(1, \frac{1}{2})$

$f_{xx}(x,y) = 6x$

$f_{xy}(x,y) = -6$

$f_{yy}(x,y) = 48y$

$H(0,0) = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$: $\det H(0,0) = -36 < 0$

$\Rightarrow T_1(0,0)$ ni lokalni ekstrem (je sedlo)

$H(1, \frac{1}{2}) = \begin{bmatrix} 6 & -6 \\ -6 & 24 \end{bmatrix}$: $\det H(1, \frac{1}{2}) = 6 \cdot 24 - 6 \cdot 6 = 36 \cdot (4-1) = 3 \cdot 36 > 0$

$f_{xx}(1, \frac{1}{2}) = 6 > 0$

$\Rightarrow T_2(1, \frac{1}{2})$ je lokalni minimum.

h) $f(x,y) = (x-y)(x+y)(y-1)$

$$f_x(x,y) = (x+y)(y-1) + (x-y)(y-1) = (y-1)(x+\cancel{y}+x-\cancel{y}) = 2x(y-1) = 0$$

$$f_y(x,y) = -(x+y)(y-1) + (x-y)(y-1) + (x-y)(x+y) =$$

$$= (y-1)\cdot(-\cancel{x}-y+\cancel{x}-y) + (x^2-y^2) =$$

$$= (y-1)(-2y) + x^2-y^2 = -2y^2+2y+x^2-y^2 = x^2-3y^2+2y = 0$$

$$2x(y-1) = 0$$

$$x^2-3y^2+2y = 0$$

1. $x=0$:

$$-3y^2+2y=0$$
$$y\cdot(2-3y)=0$$

$y_1 = 0 \qquad y_2 = \frac{2}{3}$

$T_1(0,0) \qquad T_2(0,\frac{2}{3})$

2. $x \neq 0$:

$$y=1$$
$$x^2-3+2=0$$
$$x^2=1$$

$x_1 = 1 \qquad x_2 = -1$

$T_3(1,1) \qquad T_4(-1,1)$ $\leftarrow$ stacionarne točke

$$f_{xx}(x,y) = 2(y-1)$$

$$f_{xy}(x,y) = 2x$$

$$f_{yy}(x,y) = -6y+2$$

$H(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$: $\det H(0,0) = -4 < 0$

$\Rightarrow T_1(0,0)$ ni lokalni ekstrem (je sedlo)

$H(0,\frac{2}{3}) = \begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -2 \end{bmatrix}$: $\det H(0,\frac{2}{3}) = \frac{4}{3} > 0$, $f_{xx}(0,\frac{2}{3}) < 0$

$\Rightarrow T(0,\frac{2}{3})$ je lokalni maksimum

$H(1,1) = \begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix}$: $\det H(1,1) = -4 < 0$

$\Rightarrow T(1,1)$ ni lokalni ekstrem (je sedlo)

$H(-1,1) = \begin{bmatrix} 0 & -2 \\ -2 & -4 \end{bmatrix}$: $\det H(-1,1) = -4 < 0$

$\Rightarrow T(-1,1)$ ni lokalni ekstrem (je sedlo)

i) $f(x,y) = x^2y^2$

$$\left.\begin{aligned} f_x &= 2xy^2 = 0 \\ f_y &= 2x^2y = 0 \end{aligned}\right\} \quad \begin{aligned} xy^2 &= 0 \\ x^2y &= 0 \end{aligned}$$

$xy^2 = 0 \to x = 0 \to y \in \mathbb{R}$

$xy^2 = 0 \to y = 0 \to x \in \mathbb{R}$

Stacionarne točke: $(x,0);\ x \in \mathbb{R}$
$(0,y);\ y \in \mathbb{R}$

$f_{xx} = 2y^2$
$f_{xy} = 4xy$
$f_{yy} = 2x^2$

$H(x,0) = \begin{bmatrix} 0 & 0 \\ 0 & 2x^2 \end{bmatrix}$: $\det H(x,0) = 0 \to$ nam nič ne pove

$H(0,y) = \begin{bmatrix} 2y^2 & 0 \\ 0 & 0 \end{bmatrix}$: $\det H(0,y) = 0 \to$ nam nič ne pove

$f(x,0) = 0,\ f(0,y) = 0,\ f(x,y) \geq 0$

$\Rightarrow$ $(x,0)$ in $(0,y)$ so globalni minimumi, niso pa lokalni minimumi, saj je $f(x,y) = 0$ za celotno os $x=0$ ter za celotno os $y=0$.

j) $f(x,y,z) = xyz(x+y+z-4)$ za $x,y,z>0$

dodatno (nismo naredili na vajah)

$f_x(x,y,z) = yz(x+y+z-4) + xyz = yz(2x+y+z-4) = 0$

$f_y(x,y,z) = xz(x+y+z-4) + xyz = xz(x+2y+z-4) = 0$

$f_z(x,y,z) = xy(x+y+z-4) + xyz = xy(x+y+2z-4) = 0$

$x,y,z>0 \Rightarrow$

$$\begin{aligned} 2x+y+z-4 &= 0 \\ x+2y+z-4 &= 0 \\ x+y+2z-4 &= 0 \end{aligned} \quad \begin{aligned} &\Rightarrow x=y \\ &\Rightarrow y=z \end{aligned} \quad \Rightarrow x=y=z$$

$$\begin{aligned} 2x+x+x-4 &= 0 \\ 4x &= 4 \\ x &= 1 \end{aligned}$$

Stacionarna točka $T(1,1,1)$

$f_{xx} = 2yz \qquad f_{xy} = z(2x+y+z-4)+yz$

$f_{yy} = 2xz \qquad f_{yz} = x(x+2y+z-4)+xz$

$f_{zz} = 2xy \qquad f_{xz} = y(2x+y+z-4)+yz$

$$H(1,1,1) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & \lambda-1 & 1-\lambda \end{vmatrix} = (\lambda-1)\cdot\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & -1 \end{vmatrix} =$$

$$= (\lambda-1)\cdot\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 0 & 0 & -1 \end{vmatrix} = (\lambda-1)\cdot(-1)\cdot\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} =$$

$$= -(\lambda-1)\cdot\left(\lambda^2-5\lambda+6-2\right) = -(\lambda-1)\cdot(\lambda^2-5\lambda+4) = -(\lambda-1)(\lambda-4)(\lambda-1)$$

$\Rightarrow \lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 4$

Ker so vse lastne vrednosti pozitivne, je $T(1,1,1)$ lokalni minimum.

VEZANI EKSTREMI

f... funkcija več spremenljivk

$$\left.\begin{array}{l} g_1(x_1,\dots,x_n)=a_1 \\ \quad\vdots \\ g_m(x_1,\dots,x_n)=a_m \end{array}\right\}\ \text{stranski pogoji}$$

Iščemo notranje ekstreme funkcije f pri stranskih pogojih $g_1,\dots,g_m$.

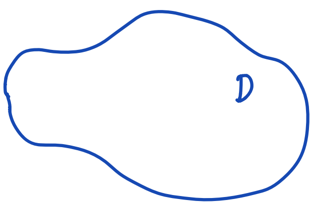
Definiramo Lagrangeovo funkcijo:

$$L(x_1,\dots,x_n,\lambda_1,\dots,\lambda_m) = f(x_1,\dots,x_n) - \lambda_1\cdot g_1(x_1,\dots,x_n) - \dots - \lambda_m\cdot g_m(x_1,\dots,x_n)$$

Notranje ekstreme funkcije f pri stranskih pogojih $g_1,\dots,g_m$ iščemo v rešitvah:

$$\begin{array}{lll} \dfrac{\partial L}{\partial x_1}=0 & \boxed{\begin{array}{c} \dfrac{\partial L}{\partial \lambda_1}=-a_1 \\ \vdots \\ \dfrac{\partial L}{\partial \lambda_m}=-a_m \end{array}} & \longrightarrow \text{raje: } \begin{array}{c} g_1=a_1 \\ \vdots \\ g_m=a_m \end{array} \\ \quad\vdots & & \\ \dfrac{\partial L}{\partial x_n}=0 & & \end{array}$$

GLOBALNI EKSTREMI



$D\subseteq\mathbb{R}^n$ omejeno, zaprto območje

f zvezna funkcija na D

Kandidati za globalne ekstreme funkcije f na D:

- lokalni ekstremi na D (iščemo stacionarne točke)
- rob (vezani ekstremi)

33. Poišči ekstrem funkcije $f(x, y) = \frac{1}{x} + \frac{1}{y}$ pri pogoju $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

$f(x,y) = \frac{1}{x} + \frac{1}{y}$

$g(x,y) = \frac{1}{x^2} + \frac{1}{y^2}$

Stranski pogoj : $g(x,y) = 1$

$L(x,y,\lambda) = f(x,y) - \lambda \cdot g(x,y) = \frac{1}{x} + \frac{1}{y} - \lambda \cdot \left(\frac{1}{x^2} + \frac{1}{y^2}\right)$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= -\frac{1}{x^2} + 2\lambda \cdot \frac{1}{x^3} = 0 \longrightarrow 2\lambda = x \\ \frac{\partial L}{\partial y} &= -\frac{1}{y^2} + 2\lambda \cdot \frac{1}{y^3} = 0 \longrightarrow 2\lambda = y \end{aligned}\right\} x = y$$

$\frac{\partial L}{\partial \lambda} = -g(x,y) = -1$

$\frac{1}{x^2} + \frac{1}{y^2} = 1$

$\frac{2}{x^2} = 1$

$x^2 = 2$

$x_1 = \sqrt{2} \quad x_2 = -\sqrt{2}$
$y_1 = \sqrt{2} \quad y_2 = -\sqrt{2}$

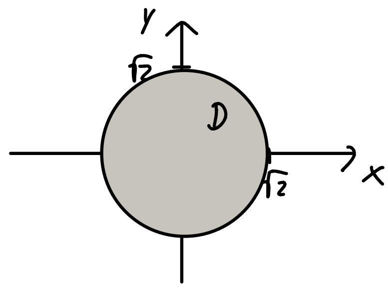
Kandidata : $T_1(\sqrt{2}, \sqrt{2})$
$T_2(-\sqrt{2}, -\sqrt{2})$

$f(\sqrt{2}, \sqrt{2}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$ maksimum ($\frac{1}{x^2} + \frac{1}{y^2} = 1$ določa omejeno območje)

$f(-\sqrt{2}, -\sqrt{2}) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$ minimum

34. Poišči maksimum in minimum funkcije $f(x,y) = x^2 + y^2 - 2y + 3$ na krogu $x^2 + y^2 \leq 2$.

LOKALNI EKSTREMI



$$\frac{\partial f}{\partial x} = 2x = 0 \rightarrow x = 0$$

$$\frac{\partial f}{\partial y} = 2y - 2 = 0 \rightarrow y = 1$$

Stacionarna točka $T(0,1) \in D$

VEZANI EKSTREMI NA ROBU

rob določen z: $x^2+y^2=2$

$g(x,y) = x^2+y^2$

$$L(x,y,\lambda) = x^2+y^2-2y+3-\lambda\cdot(x^2+y^2)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2x - 2\lambda x = 0 \longrightarrow 2x(1-\lambda) = 0 \\ \frac{\partial L}{\partial y} &= 2y - 2 - 2\lambda y = 0 \longrightarrow 2y(1-\lambda) = 2 \end{aligned}\right\}$$

1. $\underline{\lambda = 1}$
   $2x \cdot 0 = 0$ ✓
   $2y \cdot 0 = 2$ ✗
   $\Rightarrow \lambda \neq 1$

2. $\underline{\lambda \neq 1}$
   $x = 0$
   $y \cdot (1-\lambda) = 1$

$$\frac{\partial L}{\partial \lambda} = g(x,y) = 2 \rightarrow x^2+y^2 = 2 \rightarrow y^2 = 2 \rightarrow y = \pm\sqrt{2}$$

$T(0,\sqrt{2})$

$T(0,-\sqrt{2})$

Kandidati:

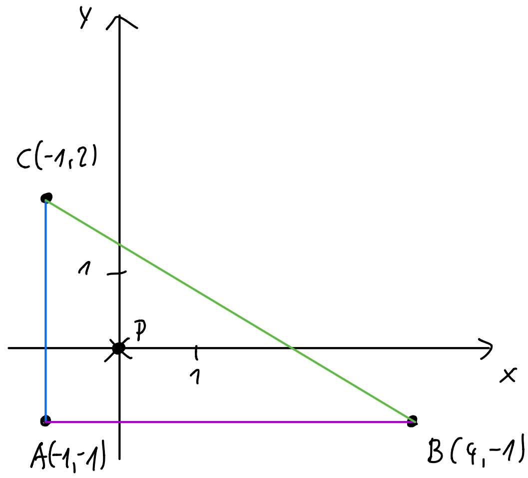
$T_1(0,1)$: $f(0,1) = 2$ minimum

$T_2(0,\sqrt{2})$: $f(0,\sqrt{2}) = 5 - 2\sqrt{2} > 2$

$T_3(0,-\sqrt{2})$: $f(0,-\sqrt{2}) = 5 + 2\sqrt{2}$ maksimum

35. Poišči maksimum in minimum funkcije $f(x,y) = e^{xy}$ na trikotniku z oglišči $A(-1,-1)$, $B(4,-1)$, $C(-1,2)$

$f(x,y) = e^{xy}$



Lokalni ekstremi:

$$\frac{\partial f}{\partial x} = e^{xy} \cdot y = 0 \quad \rightarrow y = 0$$

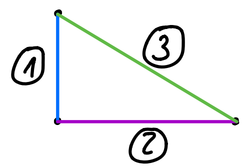
$$\frac{\partial f}{\partial y} = e^{xy} \cdot x = 0 \quad \rightarrow x = 0$$

stacionarna točka : $T(0,0) \in \Delta_{ABC}$

$f(0,0) = e^0 = 1$

Vezani ekstremi:

Rob je sestavljen iz treh delov:



1: $x = -1$

$f(-1,y) = e^{-y} = g(y)$ funkcija ene spremenljivke

$g'(y) = -e^{-y} = 0$ nikoli $\Rightarrow$ ni notranjih ekstremov na 1

$\Rightarrow$ ekstreme doseže na robo (v robnih točkah): $A(-1,-1)$ in $C(-1,2)$

2: $y = -1$

$f(x,-1) = e^{-x} = g(x)$ funkcija ene spremenljivke

$g'(x) = -e^{-x} = 0$ nikoli $\Rightarrow$ ni notranjih ekstremov na 2

$\Rightarrow$ ekstreme doseže na robo (v robnih točkah): $A(-1,-1)$ in $B(4,-1)$

3 vezani ekstrem:

stranica leži na premici z enačbo: $y = kx + n$

$$\left.\begin{array}{l} \left.\begin{array}{l} 2 = -k + n \\ -1 = 4k + n \end{array}\right\} - \quad 3 = -5k \Rightarrow k = -\frac{3}{5} \\ n = -1 - 4k = -1 + \frac{12}{5} = \frac{7}{5} \end{array}\right\} \quad y = -\frac{3}{5}x + \frac{7}{5}$$

__1. možnost:__ $y = -\frac{3}{5}\cdot x + \frac{7}{5}$

$$g(x) = f\left(x, -\tfrac{3}{5}x + \tfrac{7}{5}\right) = e^{x\cdot\left(-\frac{3}{5}x + \frac{7}{5}\right)} = e^{-\frac{3}{5}x^2 + \frac{7}{5}x}$$

$$g'(x) = e^{-\frac{3}{5}x^2 + \frac{7}{5}x} \cdot \left(-\tfrac{6}{5}\cdot x + \tfrac{7}{5}\right) = 0$$

$$-\frac{6}{5}\cdot x + \frac{7}{5} = 0 \quad /\cdot 5$$

$$6x = 7$$

$$x = \frac{7}{6}$$

$$y = -\frac{3}{5}\cdot\frac{7}{6} + \frac{7}{5} =$$

$$= \frac{7}{5}\left(-\frac{1}{2} + 1\right) = \frac{7}{10}$$

__2. možnost:__

Stranski pogoj: $y = -\frac{3}{5}\cdot x + \frac{7}{5}$ oz. $5y + 3x = 7$

$g(x,y) = 5y + 3x$

$$L(x,y,\lambda) = f(x,y) - \lambda\cdot g(x,y) =$$

$$= e^{xy} - \lambda\cdot(5y + 3x)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= y\cdot e^{xy} - 3\lambda = 0 \\ \frac{\partial L}{\partial y} &= x\cdot e^{xy} - 5\lambda = 0 \end{aligned}\right\} \quad \begin{aligned} \lambda &= \tfrac{1}{3} y e^{xy} \\ \lambda &= \tfrac{1}{5} x e^{xy} \end{aligned}$$

$$\frac{1}{3} y e^{xy} = \frac{1}{5} x e^{xy}$$

$$5y = 3x$$

$$\frac{\partial L}{\partial \lambda} = -g(x,y) = -7 \longrightarrow 5y + 3x = 7$$

$$10y = 7$$

$$y = \frac{7}{10} \qquad x = \frac{7}{6}$$

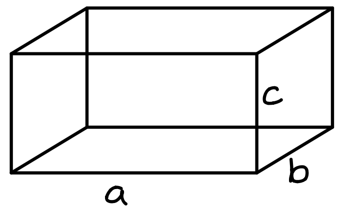
$T\left(\frac{7}{6}, \frac{7}{10}\right)$

__Vsi kandidati:__

$T(0,0)$:	$f(0,0) = 1$	
$A(-1,-1)$:	$f(-1,-1) = e$	maksimum
$B(4,-1)$:	$f(4,-1) = e^{-4}$	minimum
$C(-1,2)$:	$f(-1,2) = e^{-2}$	
$T\left(\frac{7}{6}, \frac{7}{10}\right)$:	$f\left(\frac{7}{6}, \frac{7}{10}\right) = e^{\frac{49}{60}} < e$	

36. Med kvadri s površino $P = 54$ poišči tistega z najmanjšo prostornino.

$P = 2 \cdot (a \cdot b + a \cdot c + b \cdot c)$



$V = a \cdot b \cdot c$

V ... funkcija, katere minimum iščemo

$f(a,b,c) = a \cdot b \cdot c$

$a, b, c \neq 0$

$P = 54$ ... stranski pogoj

$g(a,b,c) = 2 \cdot (ab + ac + bc)$

$L(a,b,c,\lambda) = f(a,b,c) - \lambda \cdot g(a,b,c)$

$$\left.\begin{aligned}
\frac{\partial L}{\partial a} &= b \cdot c - 2\lambda \cdot (b+c) = 0 \\
\frac{\partial L}{\partial b} &= a \cdot c - 2\lambda \cdot (a+c) = 0 \\
\frac{\partial L}{\partial c} &= a \cdot b - 2\lambda \cdot (a+b) = 0
\end{aligned}\right\}$$

$$2\lambda = \underbrace{\frac{b \cdot c}{b+c} = \frac{a \cdot c}{a+c}}_{1} = \frac{a \cdot b}{a+b}, \qquad \frac{a \cdot c}{a+c} = \frac{a \cdot b}{a+b} \ \ (2)$$

1: $b \cdot (a+c) = a \cdot (b+c)$
$\cancel{ba} + bc = \cancel{ab} + ac$
$bc = ac$
$b = a$

2: $c \cdot (a+b) = b \cdot (a+c)$
$ca + \cancel{cb} = ba + \cancel{bc}$
$ca = ba$
$c = b$

$\Rightarrow a = b = c$

$$\frac{\partial L}{\partial \lambda} = \underbrace{-g(a,b,c) = -54}_{g(a,b,c) = 54}$$

$2 \cdot (ab + ac + bc) = 54$

$ab + ac + bc = 27$

$3a^2 = 27$

$a^2 = 9$

$a = 3$

$\underline{\underline{a = b = c = 3}}$

* Bolj splošno

$g(a,b,c) = P \in \mathbb{R}$

$\frac{\partial L}{\partial a}, \frac{\partial L}{\partial b}, \frac{\partial L}{\partial c}$ isti kot prej $\Rightarrow a = b = c$

$3a^2 = P \Rightarrow a = \sqrt{\frac{P}{6}}$

17.11.2020

37. Med trikotniki z obsegom o poišči tistega z največjo ploščino.

$o = a+b+c$ ... stranski pogoj : $g(a,b,c) = a+b+c$

$g(a,b,c) = o$

$P = f(a,b,c) = \sqrt{s\cdot(s-a)\cdot(s-b)\cdot(s-z)}$; $s = \frac{a+b+c}{2}$ ... funkcija, katere maksimum iščemo

$L(a,b,c,\lambda) = f(a,b,c) - \lambda\cdot g(a,b,c)$

$$\frac{\partial L}{\partial a} = \frac{1}{2}\cdot\frac{1}{\sqrt{s\cdot(s-a)(s-b)\cdot(s-c)}}\cdot\left(s\cdot(-1)\cdot(s-b)\cdot(s-c)\right)\cdot\underbrace{1}_{\frac{\partial s}{\partial a}} - \lambda = 0$$

$$\frac{\partial L}{\partial a} = \frac{1}{2}\cdot\frac{1}{\sqrt{s\cdot(s-a)(s-b)\cdot(s-c)}}\cdot\left(s\cdot(s-a)\cdot(-1)\cdot(s-c)\right)\cdot 1 - \lambda = 0$$

$$\frac{\partial L}{\partial a} = \frac{1}{2}\cdot\frac{1}{\sqrt{s\cdot(s-a)(s-b)\cdot(s-c)}}\cdot\left(s\cdot(s-a)\cdot(s-b)\cdot(-1)\right)\cdot 1 - \lambda = 0$$

$-\frac{2\lambda}{s}\cdot\sqrt{s\cdot(s-a)(s-b)\cdot(s-c)} =$

$= (s-b)\cdot(s-c) =$

$= (s-a)\cdot(s-c) =$

$= (s-a)\cdot(s-b)$

$\Downarrow$

$a = b = c$

$\frac{\partial L}{\partial \lambda} = -g(a,b,c) = -o \longrightarrow g(a,b,c) = o$

$$a+b+c = o$$

$$3a = o$$

$$a = \frac{o}{3} \Rightarrow \underline{\underline{a = b = c = \frac{o}{3}}}$$

*Opomba 1: pri zapisu predpisa funkcije f lahko že upoštevamo, da je $s = \frac{o}{2}$ konstantna.

Opomba 2: Ekstremi funkcije $\sqrt{s\cdot(s-a)\cdot(s-b)\cdot(s-z)}$ so enaki ekstremom funkcije $s\cdot(s-a)\cdot(s-b)\cdot(s-z)$, zato lahko za f vzamemo predpis

$f(a,b,c) = s\cdot(s-a)\cdot(s-b)\cdot(s-z)$

38. Na elipsoidu $\frac{x^2}{96}+y^2+z^2=1$ poišči točko, ki je najbolj oddaljena od ravnine $3x+4y+12z=35$.

oddaljenost točke (x_0,y_0,z_0) od ravnine z enačbo $ax+by+cz=e$:

$$d=\frac{|ax_0+by_0+cz_0-e|}{\sqrt{a^2+b^2+c^2}}$$

Ravnina: $3x+4y+12z=35$
Točka : (x,y,z)

$$f(x,y,z)=\frac{|3x+4y+12z-35|}{\sqrt{9+16+144}}=\frac{|3x+4y+12z-35|}{\sqrt{169}}=\frac{|3x+4y+12z-35|}{13}$$

Stranski pogoj: $g(x,y,z)=\frac{x^2}{96}+y^2+z^2=1$

$$L(x,y,z,\lambda)=f(x,y,z)-\lambda\cdot g(x,y,z)$$

1.) $3x+4y+12z-35\geq 0$:

$$L(x,y,z,\lambda)=\frac{1}{13}\cdot(3x+4y+12z-35)-\lambda\cdot\left(\frac{x^2}{96}+y^2+z^2\right)$$

$$\frac{\partial L}{\partial x}=\frac{3}{13}-\frac{2\lambda}{96}x=0\longrightarrow x=\frac{3}{13}\cdot\frac{96}{2\lambda}$$

$$\frac{\partial L}{\partial y}=\frac{4}{13}-2\lambda y=0\longrightarrow y=\frac{4}{13}\cdot\frac{1}{2\lambda}$$

$$\frac{\partial L}{\partial z}=\frac{12}{13}-2\lambda z=0\longrightarrow z=\frac{12}{13}\cdot\frac{1}{2\lambda}$$

2.) $3x+4y+12z-35<0$:

$$L(x,y,z,\lambda)=-\frac{1}{13}\cdot(3x+4y+12z-35)-\lambda\cdot\left(\frac{x^2}{96}+y^2+z^2\right)$$

$g(x,y,z)=1$: $\quad \frac{x^2}{96}+y^2+z^2=\frac{3^2}{13^2}\cdot\frac{96^2}{(2\lambda)^2}\cdot\frac{1}{96}+\frac{4^2}{13^2}\cdot\frac{1}{(2\lambda)^2}+\frac{12^2}{13^2}\cdot\frac{1}{(2\lambda)^2}=1\quad\Big/\cdot 13^2\cdot(2\lambda)^2$

$$9\cdot 96+16+12^2=13^2\cdot 4\cdot\lambda^2\quad /:4$$

$$9\cdot 24+4+12\cdot 3=13^2\lambda^2$$

$$256=13^2\lambda^2$$

$$16^2=13^2\lambda^2$$

$$\lambda=\pm\frac{16}{13}$$

$\Rightarrow$

$$x_1=\frac{3}{\cancel{13}}\cdot\frac{96}{2}\cdot\frac{\cancel{13}}{16}=\frac{3\cdot\cancel{16}\cdot 3\cdot\cancel{2}}{\cancel{2}\cdot\cancel{16}}=9$$

$$y_1=\frac{\cancel{4}}{\cancel{13}}\cdot\frac{1}{2}\cdot\frac{\cancel{13}}{\cancel{16}_4}=\frac{1}{8}$$

$$z_1=\frac{12}{\cancel{13}}\cdot\frac{1}{2}\cdot\frac{\cancel{13}}{16}=\frac{3}{8}$$

$$x_2=-9,\ y_2=-\frac{1}{8},\ z_2=-\frac{3}{8}$$

kandidata: $T_1\left(9, \frac{1}{8}, \frac{3}{8}\right)$:

$$f\left(9, \frac{1}{8}, \frac{3}{8}\right) = \frac{1}{13} \cdot \left|3 \cdot 9 + 4 \cdot \frac{1}{8} + 12 \cdot \frac{3}{8} - 35\right| =$$

$$= \frac{1}{13} \cdot \left|27 + \frac{1}{2} + \frac{9}{2} - 35\right| =$$

MINIMUM

$$= \frac{1}{13}\left|-8 + \frac{10}{2}\right| = \frac{3}{13}$$

$T_2\left(-9, -\frac{1}{8}, -\frac{3}{8}\right)$:

$$f\left(-9, -\frac{1}{8}, -\frac{3}{8}\right) = \frac{1}{13} \cdot \left|-27 - 5 - 35\right| =$$

MAKSIMUM

$$= \frac{1}{13} \cdot 67 = \frac{67}{13}$$

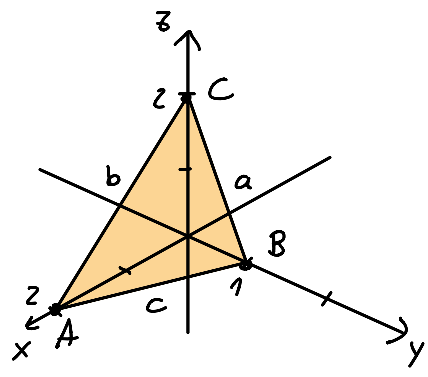
Točka $T_2\left(-9, -\frac{1}{8}, -\frac{3}{8}\right)$ je izmed točk na elipsoidu $\frac{x^2}{96} + y^2 + z^2 = 1$ najbolj oddaljena od ravnine $\Sigma(3x + 4y + 12z = 35)$.

Funkcija f je podana s predpisom

$$f(x,y,z) = x^2 + y^2 + z^2.$$

Poišči globalne ekstreme funkcije f na trikotniku z enačbo $x+2y+z=2$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

EKSTREMI V NOTRANJOSTI TRIKOTNIKA:



to so ekstremi funkcije $f(x,y,z)$ na $\overbrace{\text{ravnini z enačbo } x+2y+z=2}^{\text{pogoj}}$
(na koncu preverimo, ali za njih hkrati velja $x \geq 0, y \geq 0, z \geq 0$).

$L(x,y,z,\lambda) = f(x,y,z) - \lambda \cdot g(x,y,z)$; $\quad g(x,y,z) = x+2y+z$

$$\left.\begin{aligned} L_x &= 2x - \lambda = 0 \\ L_y &= 2y - 2\lambda = 0 \\ L_z &= 2z - \lambda = 0 \end{aligned}\right\} \begin{aligned} &\lambda = 2x = y = 2z \\ &\Rightarrow \; z = x,\; y = 2x \end{aligned}$$

$g(x,y,z) = 2$: $\quad x + 2y + z = 2$

$x + 4x + x = 2$

$6x = 2 \qquad y = 2x \qquad z = x$

$x = \frac{1}{3} \qquad y = \frac{2}{3} \qquad z = \frac{1}{3} \qquad x \geq 0, y \geq 0, z \geq 0$ ✓

Kandidati v notranjosti: $T(\frac{1}{3}, \frac{2}{3}, \frac{1}{3})$

EKSTREMI NA ROBU:

1.) STRANICA a: $\qquad x=0,\; x+2y+z=2 \qquad (0 \leq y \leq 1)$

Opazujemo ekstreme funkcije $f(x,y,z)$ na premici $x=0, x+2y+z=2$.

$$\left.\begin{aligned} x &= 0 \\ x+2y+z &= 2 \end{aligned}\right\} \begin{aligned} x &= 0 \\ z &= 2-2y \end{aligned} \quad \Rightarrow \quad f(x,y,z) = f(0,y,2-2y) = y^2 + (2-2y)^2$$

$g(y) = y^2 + (2-2y)^2 = y^2 + 4 - 8y + 4y^2 = 5y^2 - 8y + 4$

$g'(y) = 10y - 8 = 0 \Rightarrow y = \frac{4}{5},\; z = 2-2y = 2 - \frac{8}{5} = \frac{2}{5},\; x = 0$

na stranici ker $0 \leq \frac{4}{5} \leq 1$

kandidati za ekstreme: $T(0, \frac{4}{5}, \frac{2}{5})$, B, C

2.) STRANICA b: $\quad y=0,\ x+2y+z=2 \quad (0\le x\le 2)$

Opazujemo ekstreme funkcije $f(x,y,z)$ na premici $y=0, x+2y+z=2$.

$$\left.\begin{array}{l} y=0 \\ x+2y+z=2 \end{array}\right\} \begin{array}{l} y=0 \\ z=2-x \end{array} \quad \Rightarrow \quad f(x,y,z)=f(x,0,2-x)=x^2+(2-x)^2$$

$$g(x)=x^2+(2-x)^2=x^2+4-4x+x^2=2x^2-4x+4$$

$$g'(x)=4x-4=0 \Rightarrow x=1,\ y=0,\ z=2-1=1$$

na stranici, ker $0\le 1\le 2$

kandidati za ekstreme: $T(1,0,1), A, C$

3.) STRANICA c: $\quad z=0,\ x+2y+z=2 \quad (0\le y\le 1)$

Opazujemo ekstreme funkcije $f(x,y,z)$ na premici $z=0, x+2y+z=2$.

$$\left.\begin{array}{l} z=0 \\ x+2y+z=2 \end{array}\right\} \begin{array}{l} z=0 \\ x=2-2y \end{array} \quad \Rightarrow \quad f(x,y,z)=f(2-2y,y,0)=(2-2y)^2+y^2$$

$$g(y)=5y^2-8y+4$$

$$g'(y)=10y-8=0 \Rightarrow y=\frac{4}{5},\ x=\frac{2}{5},\ z=0$$

kandidati za ekstreme: $\underbrace{T\left(\frac{2}{5},\frac{4}{5},0\right)}_{\text{na stranici ker } 0\le \frac{4}{5}\le 1}, A, B$

VSI KANDIDATI ZA EKSTREME:

$T\left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)$: $\quad f\left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)=\frac{1}{9}+\frac{4}{9}+\frac{1}{9}=\frac{6}{9}=\frac{2}{3}$ $\leftarrow$ MINIMUM

$T\left(0,\frac{4}{5},\frac{2}{5}\right)$: $\quad f\left(0,\frac{4}{5},\frac{2}{5}\right)=\frac{16}{25}+\frac{4}{25}=\frac{20}{25}=\frac{4}{5}$

$T(1,0,1)$: $\quad f(1,0,1)=1+1=2$

$T\left(\frac{2}{5},\frac{4}{5},0\right)$: $\quad f\left(\frac{2}{5},\frac{4}{5},0\right)=\frac{4}{5}$

$A(2,0,0)$: $\quad f(2,0,0)=4$ $\leftarrow$ MAKSIMUM

$B(0,1,0)$: $\quad f(0,1,0)=1$

$C(0,0,2)$: $\quad f(0,0,2)=4$ $\leftarrow$ MAKSIMUM

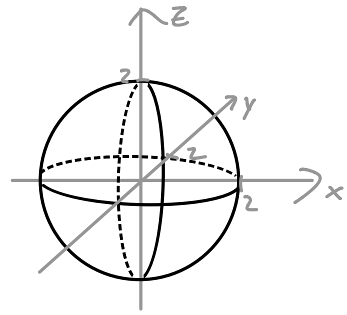
40. V krogli

$$x^2 + y^2 + z^2 \leq 4$$

je gostota porazdeljena s predpisom $\rho(x, y, z) = 3 + xz + y^2$. Kje v krogli je gostota največja in kje najmanjša? Kolikšni sta ekstremni vrednosti?

EKSTREMI V NOTRANJOSTI:

$\rho(x,y,z) = 3 + xz + y^2$



$$\left.\begin{aligned} \rho_x(x,y,z) &= z = 0 \\ \rho_y(x,y,z) &= 2y = 0 \\ \rho_z(x,y,z) &= x = 0 \end{aligned}\right\} \quad T(0,0,0)$$

: ker leži v notranjosti krogle, je kandidat za ekstrem

EKSTREMI NA ROBU:

rob je določen z enačbo $\underbrace{x^2+y^2+z^2=4}_{\text{stranski pogoj}}$ $\Rightarrow g(x,y,z) = x^2+y^2+z^2$, $g(x,y,z) = 4$

$$L(x,y,z,\lambda) = \rho(x,y,z) - \lambda \cdot g(x,y,z) = 3 + xz + y^2 - \lambda\cdot(x^2+y^2+z^2)$$

$$\left.\begin{aligned} L_x &= z - 2\lambda x = 0 \\ L_y &= 2y - 2\lambda y = 0 \\ L_z &= x - 2\lambda z = 0 \end{aligned}\right\} \quad 2y\cdot(1-\lambda) = 0$$

$y = 0$: $z = 2\lambda x$, $x = 2\lambda z$

$z = (2\lambda)^2 z$

$z\cdot(1-4\lambda^2) = 0$

$\lambda = 1$: $z - 2x = 0$, $x - 2z = 0 \quad /\cdot 2$, $+$

$z - \cancel{2x} + \cancel{2x} - 4z = 0$

$\rightarrow z = 0 \rightarrow x = 0$, $y \in \mathbb{R}$

$z = 0 \rightarrow x = 0$

$4\lambda^2 = 1 \rightarrow \lambda = \pm\frac{1}{2}$

$z = \pm 2\cdot\frac{1}{2}\cdot x$

$x = \pm 2\cdot\frac{1}{2} z$

$\rightarrow z = \pm x$

$g(x,y,z) = 4$:

$0^2 + y^2 + 0^2 = 4 \Rightarrow y = \pm 2$

$T(0,2,0)$, $T(0,-2,0)$

$0^2 + 0^2 + 0^2 = 4$

$\nrightarrow$

$x^2 + 0^2 + (\pm x)^2 = 4$

$2x^2 = 4 \rightarrow x = \pm\sqrt{2}$

$T(\sqrt{2},0,\sqrt{2})$, $T(-\sqrt{2},0,-\sqrt{2})$,
$T(\sqrt{2},0,-\sqrt{2})$, $T(-\sqrt{2},0,\sqrt{2})$

VSI KANDIDATI:

$T(0,0,0)$: $\rho(0,0,0) = 3$

$T(0,2,0)$, $T(0,-2,0)$: $\rho(0,\pm 2,0) = 7$ MAKSIMUM

$T(\sqrt{2},0,\sqrt{2})$, $T(-\sqrt{2},0,-\sqrt{2})$: $\rho(\pm\sqrt{2},0,\pm\sqrt{2}) = 5$

$T(\sqrt{2},0,-\sqrt{2})$, $T(-\sqrt{2},0,\sqrt{2})$: $\rho(\pm\sqrt{2},0,\mp\sqrt{2}) = 1$ MINIMUM