$$\sqrt[n]{n} \xrightarrow{n \to \infty} 1$$

$\sqrt[n]{n} > 1$ za vsak $n \in \mathbb{N}$

Za vsak $\varepsilon > 0$ obstaja $N \in \mathbb{N}$, da je za vse $n \geq N$:

$$\underline{1 < \sqrt[n]{n} < 1 + \varepsilon}$$

$$\sqrt[n]{n} < 1 + \varepsilon$$

$$n < \underline{(1+\varepsilon)^n}$$

za $n \geq N$ od nekega N naprej

$$(1+\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} \varepsilon^k = \qquad \begin{matrix}\wedge \\ \shortparallel\end{matrix}$$

$$= \underline{1 + n \cdot \varepsilon + \binom{n}{2}\varepsilon^2 + \dots + \binom{n}{n}\varepsilon^n}$$

$> n$, za dovolj velike n

$> n$

$$n \geq \underline{1 + n \cdot \varepsilon + \binom{n}{2}\varepsilon^2 + \dots + \binom{n}{n}\varepsilon^n}$$

$$n > \binom{n}{2}\varepsilon^2$$ za vse n

↳ ne velja

$$\binom{n}{2} = \frac{n(n-1)}{2}\varepsilon^2$$

$$n > \frac{n(n-1)}{2}\varepsilon^2$$ za vse n

$$1 > \underbrace{(n-1)}_{\downarrow\, n\to\infty \,\atop \infty}\ \underbrace{\frac{\varepsilon^2}{2}}_{\hookrightarrow\, >0}$$ za vse n

$\ast$

$$\exists N : \ (n-1)\frac{\varepsilon^2}{2} \geq 1$$

za dovolj velike n:

$$n < (1+\varepsilon)^n$$

$$\sqrt[n]{n} < (1+\varepsilon)$$

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \longrightarrow 0$$

$$\begin{tikzcd}[column sep=small]
1 < & \sqrt[n]{n} \arrow[d] & < & (1+\varepsilon_n) \arrow[d] \\
 & 1 & & 1
\end{tikzcd}$$

$$\sqrt[n]{\underbrace{n+1-1}} = (n+1-1)^{\frac{1}{n}} = \quad ?$$

$$= \left(1+\left(\frac{1}{n-1}\right)^{\boxed{-1}}\right)^{\frac{1}{n}}$$

$$\lim (1+n)^{\frac{1}{n}} \qquad \hookrightarrow \frac{1}{\frac{1}{n-1}}$$

$$\sqrt[n]{n+1-1} = (n+1-1)^{\frac{1}{n}} = \qquad \curvearrowright \quad t := \frac{1}{n-1}$$

$$= \left(1+(n-1)\right)^{\frac{1}{n}} = \qquad t(n-1) = 1$$

$$n-1 = \frac{1}{t}$$

$$= \left(1+\frac{1}{t}\right)^{\frac{t}{1-t}} = \qquad n = \frac{1}{t}-1$$

$$n = \frac{1-t}{t}$$

$$= \left[\left(1+\frac{1}{t}\right)^{t}\right]^{\frac{1}{1-t}} \qquad \frac{1}{n} = \frac{t}{1-t}$$

$$\Big\downarrow t \to 0$$

$$1$$

Številske vrste

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt[3]{n}} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}\left(\sqrt{n+1} + \sqrt{n}\right)}$$

$$\hookrightarrow \overset{\leq}{\approx} \frac{1}{\sqrt[3]{n}\left(\sqrt{n} + \sqrt{n}\right)} =$$

$$= \frac{1}{2\sqrt[3]{n} \cdot \sqrt{n}} =$$

$$= \frac{1}{2} \frac{1}{n^{\frac{1}{2}+\frac{1}{3}}} =$$

$$= \frac{1}{2}\left(\frac{1}{n}\right)^{\frac{5}{6}}$$

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n}\right)^{5/6}$$

iščemo vrsto s členi $0 \le a_n \le \frac{1}{\sqrt[3]{n}\left(\sqrt{n+1}+\sqrt{n}\right)}$,

$\sum_{n=1}^{\infty} a_n$ div.

$$a_n = \frac{1}{\sqrt[3]{n+1}\left(\sqrt[2]{n+1}\cdot 2\right)} \le \frac{1}{\sqrt[3]{n}\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)}\right)^{\frac{5}{6}}$, divergira

Sklep: vrsta divergira

$$\sum_{n=1}^{\infty}\frac{n^n}{(n^2)!}$$

korenski : $\sqrt[n]{\frac{n^n}{(n^2)!}} = \frac{\overset{\to \infty}{n}}{\underset{\to \infty}{\sqrt[n]{(n^2)!}}} \longrightarrow R < 1$

$$(n^2)! = \underbrace{n^2(n^2-1)(n^2-2)\dots}_{\substack{\text{vsi členi} > n, \\ \#\text{ členov } n^2-n}} \underbrace{n\cdot(n-1)\cdot\dots\cdot 1}_{>1,\ = n!}$$

$\Rightarrow (n^2)! > n^n \cdot n! \Rightarrow (n^2)! > n\cdot\sqrt[n]{n!}$

$$\frac{n}{\sqrt[n]{(n^2)!}} < \frac{\cancel{n}}{\cancel{n}\cdot\underbrace{\sqrt[n]{n!}}_{\infty}} \xrightarrow{n\to\infty} 0 < 1$$

Sklep: vrsta konvergira

$(a_n)_n$ in $(b_n)_n$ zaporedji, za kateri velja:

$$\sum_{n=0}^{\infty} a_n^2,\ \sum_{n=0}^{\infty} b_n^2 < \infty$$

Dokaži, da absolutno konvergira zaporedje $\sum_{n=0}^{\infty} a_n b_n$

$$|a_n b_n| < a_n^2,\ b_n^2$$

[če vrsta konvergira, so njeni členi po abs. vrednosti od nekega n naprej < 1]

če $|a_n| \geq |b_n|$, potem je

$$|a_n b_n| \leq |a_n|^2$$

če $|b_n| > |a_n|$, potem je

$$|a_n b_n| < |b_n|^2$$

∞

$$\sum_{n=1}^{\infty} |a_n b_n| = \sum_{\substack{n=1\\|a_n|\geq|b_n|}}^{\infty} |a_n b_n| + \sum_{\substack{n=1\\|a_n|<|b_n|}}^{\infty} |a_n b_n|$$

$$\leq \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 < \infty$$

Torej tudi ta vrsta konvergira.

ALTERNIRAJOČE VRSTE

$a_n \geq 0$; sod n (ali obratno)
$a_n \leq 0$; lih n

vrsta konv. absolutno => vrsta konvergira
vrsta konv., ni konv. absolutno => vrsta pogojno konvergira

LEIBNIZOV KRITERIJ ZA ALT. VRSTE:

$|a_n| > |a_{n+1}| > |a_{n+2}| > \dots$

$|a_n| \xrightarrow{n \to \infty} 0$, potem vrsta konvergira

OBRAVNAVAJ KONVERGENCO VRSTE

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[3]{n-1}}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{1}{3}} \quad (n-1 \sim n)$$ divergira,

zato $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[3]{n-1}}$ absolutno ne konvergira

$$\underset{|a_n|}{\underset{\shortparallel}{\frac{1}{\sqrt[3]{n-1}}}} > \underset{|a_{n+1}|}{\underset{\shortparallel}{\frac{1}{\sqrt[3]{n}}}} \quad \checkmark$$

$$\frac{1}{\sqrt[3]{n-1}} \xrightarrow{n \to \infty} 0$$

Sledi: vrsta pogojno konvergira

$\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n+3^n}}_{*}$ ... obravnavaj konvergenco vrste

$2^n+3^n > 2^n+2^n$

$\swarrow (\ )^{-1}$

$$\frac{1}{2^n+3^n} < \frac{1}{2\cdot 2^n}$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$$

Torej tudi vrsta * konvergira

$$\sum_{n=2}^{\infty} \frac{(a+7)^n}{15^n \cdot \ln n} \to a_n$$

Vprašanje: za katere $a \in \mathbb{R}$ vrsta konvergira?

$\left|\frac{a+7}{15}\right| < 1$, vrsta konvergira

V tem primeru je $|a_n| < \left|\frac{a+7}{15}\right|^n$.

$\sum_{n=2}^{\infty}\left|\frac{a+7}{15}\right|^n$ konvergira $\Rightarrow \sum_{n=2}^{\infty} a_n$ absolutno konvergira

$\left|\frac{a+7}{15}\right| > 1$, vrsta divergira

$\cancel{\sum_n}$ $$\frac{\left|\frac{a+7}{15}\right|^n}{\ln n} > \frac{\left|\frac{a+7}{15}\right|}{\ln n}$$

Vrsta $\sum_{n=2}^{\infty} \left|\frac{a+7}{15}\right| \cdot \frac{1}{\ln n}$ divergira

$\Rightarrow$ $\sum_{n=2}^{\infty} |a_n|$ divergira

$$\sum_{n=2}^{\infty} a_n$$

$c > 1$, $\frac{c^n}{\ln n} > \frac{c^n}{n}$

$$\begin{tikzcd} \frac{c^n}{n} \arrow[d, "\text{korenski kriterij}"] \\ \text{divergira} \end{tikzcd}$$

$$\frac{|c|^n}{n} \xrightarrow{n \to \infty} \infty$$

(korenite)

$$\frac{c^n}{n} \cancel{\xrightarrow{n \to \infty}} 0$$

Torej vrsta divergira (ne le absolutno).

$$\left|\frac{a+7}{15}\right| = 1$$

- $\frac{a+7}{15} = 1$, $a_n = \frac{1}{\ln n}$,

vrsta divergira

- $\frac{a+7}{15} = -1$, $a_n = \frac{(-1)^n}{\ln n}$

vrsta abs. ne konvergira

$$\frac{1}{\ln n} \xrightarrow{n \to \infty} 0$$

$$\frac{1}{\ln n} > \frac{1}{\ln(n+1)}$$

Po Leibnizovem kriteriju vrsta konvergira.

Vrsta absolutno konvergira, če je

$$\left|\frac{a+7}{15}\right| < 1$$

$$|a+7| < 15 \iff a \in (-22, 8)$$

Vrsta divergira, če je

$$\left|\frac{a+7}{15}\right| > 1$$

$$|a+7| > 15 \iff a \in (-\infty, -22) \cup [8, \infty)$$

Vrsta pogojno konvergira, či je $a = -22$

$\sum_{n=1}^{\infty} a_n$, konvergentna, členi so pozitivni

$\sum_{n=1}^{\infty} a_n^2$ konvergira $\quad *$

Namig: $\sum a_n < \infty \Rightarrow \lim_{n\to\infty} a_n = 0$

$$\Rightarrow a_n < 1 \Rightarrow a_n^2 < a_n$$

$$\Downarrow$$

$$a_n > 0$$

$\hookrightarrow$ za vse n od nekega $N \in \mathbb{N}$ naprej

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N} a_n^2 + \sum_{n=N}^{\infty} a_n^2 \Rightarrow$$

zaporedje konvergira

$$\sum_{n=1}^{N} a_n^2 \; \wedge \; \infty$$

$$\sum_{n=N}^{\infty} a_n^2 \; \wedge \; \sum_{n=N}^{\infty} a_n \; \wedge \; \infty$$

Ali lahko sklepamo obratno? $a_n \geq 0$

$$\sum a_n^2 < \infty \Rightarrow \sum a_n < \infty$$

$$a_n = \frac{1}{n}$$

$\sum a_n^2$ konvergira, $\sum a_n$ divergira

a_n niso vsi pozitivni, ali velja $*$?

$$a_n = \frac{(-1)^n}{\sqrt{n}}$$

$\sum a_n$ konvergira po Leibnizovem kriteriju,

$\sum a_n^2 = \sum \frac{1}{n}$ pa divergira.

$(a_n)_n$ padajoče zaporedje pozitivnih števil.

$$\sum_{n=1}^{\infty} a_n < \infty \iff \sum_{k=0}^{\infty} 2^k a_{2^k} < \infty$$

$\Leftarrow$

$\sum_{k=0}^{\infty} 2^k a_{2^k}$:

k

$$\underbrace{a_1}_{k=0} + \underbrace{a_2 + \overset{a_3}{\overset{\wedge}{a_2}}}_{k=1} + \underbrace{a_4 + \overset{a_5}{\overset{\wedge}{a_4}} + \overset{a_6}{\overset{\wedge}{a_4}} + \overset{a_7}{\overset{\wedge}{a_4}}}_{k=2} +$$

$$+ \underbrace{a_8 + \overset{a_9}{\overset{\wedge}{a_8}} + \dots + \overset{a_{15}}{\overset{\wedge}{a_8}}}_{k=3} + a_{16} \cdot 16 + \dots$$

$$\sum_{k=0}^{\infty} 2^k a_{2^k} \geq \sum_{n=1}^{\infty} a_n > 0$$

Po primerjalnem kriteriju konvergira $\sum_{n=1}^{\infty} a_n$.

$$\Rightarrow \quad \sum a_n < \infty$$

$$\frac{1}{2}\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^{k-1} a_{2^k}$$

$$\begin{array}{ccccccc}
\frac{1}{2} a_1 & + & a_2 & + & \underbrace{a_4 + a_4}_{k=2} & + & \underbrace{a_8 + a_8 + \dots + a_8}_{k=3} \; + \\
k=0 & & k=1 & & & & \\
\rotatebox{90}{$\geq$} & & & & \rotatebox{90}{$\geq$} \qquad & & \rotatebox{90}{$\geq$} \qquad\qquad \rotatebox{90}{$\geq$} \\
a_1 & & & & a_3 \qquad & & a_5 \qquad\qquad a_8
\end{array}$$

$$+ \dots$$

Podobno kot zgoraj sklepamo:

$$\frac{1}{2}\sum_{k=0}^{\infty} 2^k a_{2^k} \leq \sum_{n=1}^{\infty} a_n$$

✓

$$\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$$ ... ali ta vrsta konvergira?

$$\sum_{k=1}^{\infty} \frac{2^k}{2^k \cdot (\ln 2^k)}$$ ... ali ta vrsta konvergira?

$$= \sum_{k=1}^{\infty} \frac{1}{k \cdot \ln 2} = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{k}$$

↳ divergira

Sklep: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ divergira.

$$\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2}$$ ... ali ta vrsta konvergira?

$$\sum_{k=1}^{\infty} \frac{2^4}{2^k (\ln 2^k)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \frac{1}{(\ln 2)^2}$$

✓

Vrsta konvergira.

$$\frac{1}{n} > \frac{1}{n \ln n} > \left(\frac{1}{n}\right)^s, \quad s > 1$$

od nekega n naprej

Seštej neskončno vsoto

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{5^n} = *$$

$$\cos(n\pi) = (-1)^n$$

$$* = \sum_{n=1}^{\infty} \frac{(-1)^n}{5^n} =$$

$$= \sum_{k=1}^{\infty} \frac{1}{5^{2k}} - \sum_{k=1}^{\infty} \frac{1}{5^{2k-1}} =$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{25}\right)^k - 5 \sum_{k=1}^{\infty} \left(\frac{1}{25}\right)^k =$$

$$= (-4) \sum_{k=1}^{\infty} \frac{1}{25^k} =$$

$$= (-4) \frac{\frac{1}{25}}{1 - \frac{1}{25}} = \dots$$

FUNKCIJE

$$f : D_f \longrightarrow \mathbb{R}$$

D_f ... definicijsko območje

$\cap$

$\mathbb{R}$

$$f(D_f) = Z_f = \{ f(x) \mid x \in D_f \}$$

FUNKCIJE IMAJO RAZLIČNA DEF. OBMOČJA, ZATO SO LAHKO RAZLIČNE, TUDI ČE JE IZRAZ ENAK

$f(x) = |x|$; $D_f = \mathbb{R}$

$g(x) = (\sqrt[2]{x})^2$ $D_g = \mathbb{R}_{\geq 0}$

f je injektivna, če je $f(x) \neq f(y)$ za $x, y \in D_f$, $x \neq y$

f je surjektivna, če je $\underline{Z_f = \mathbb{R}}$

$|z|$ za vsak $z \in \mathbb{C}$; ni surj.

$f : [0, \infty) \longrightarrow [0, \infty)$, $f(x) = |x|$; je surj.

$f: A \longrightarrow B$; $g: B \longrightarrow C$

$g \circ f$ injektivna.

$\underline{f \text{ injektivna}}$

Recimo: $f(x_1) = f(x_2)$ za $x_1 \neq x_2$, $\in A$

$\Rightarrow g \circ f(x_1) = g \circ f(x_2)$

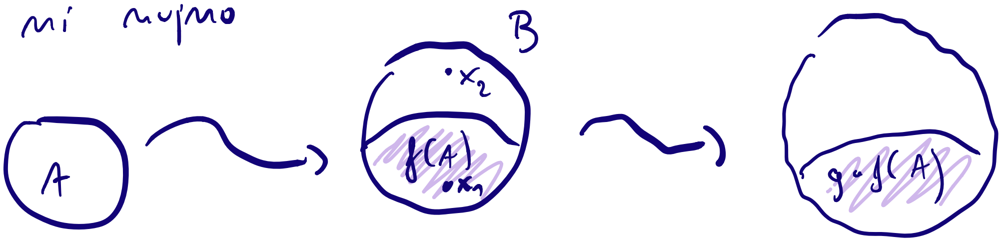
$\times$

$g \circ f$ injektivna

Sledi: f je injektivna.

g injektivna?

Ideja: ni nujno



$g(x_2) = g(x_1)$

$g \circ f$ injektivna $\not\Rightarrow$ g injektivna

$f(x) = \sqrt[2]{x}$

$g(x) = x^2$

$(\sqrt[2]{x})^2 = g \circ f(x)$, definirana na $[0, \infty)$, je injektivna

$g \circ f$ surjektivna $\Rightarrow$ g surjektivna, f ni nujno surjektivna

SODOST / LIHOST FUNKCIJ

$f : \mathbb{R} \longrightarrow \mathbb{R}$ je SODA, če je $f(x) = f(-x)$ za vse $x \in \mathbb{R}$

$f : \mathbb{R} \longrightarrow \mathbb{R}$ je LIHA, če je $f(x) = -f(-x)$ za vse $x \in \mathbb{R}$

Za vsako funkcijo f obstajata funkciji f_s (soda) in f_l (liha), da velja

$$\underline{f(x) = f_s(x) + f_l(x)}$$

$$2 f(x) = f(x) - f(-x) + f(x) + f(-x)$$

$$f(x) = \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{liha}} + \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{soda}}$$