Polarni zapis kompleksnih števil - nadaljevanje

$$z = r(\cos(\varphi) + i\sin(\varphi))$$

$r = |z|$; φ - kot - argument

De Moivreova formula

$$z^n = |z|^n(\cos(n\varphi) + i\sin(n\varphi))$$

Korenjenje kompleksnih števil:

$\sqrt[n]{z} = w$; $w^n = z$

$$w = s(\cos(\psi) + i\sin(\psi))$$

Veljati mora:

$$w^n = z$$

torej

$$s^n(\cos(n\psi) + i\sin(n\psi)) = r(\cos(\varphi) + i\sin(\varphi))$$

Torej:

$$s = \sqrt[n]{r}$$

$$\psi = \frac{\varphi}{n}$$

$$w = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n}\right) + i\sin\left(\frac{\varphi}{n}\right)\right)$$

Velja: $w^n = z$

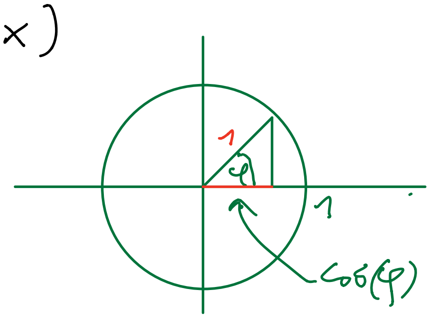
Toda, to ni edini n-ti koren števila z.

Spomniti se moramo periodičnosti funkcij cos in sin:

Velja: za vsak x imamo

$$\cos(x + 2\pi) = \cos(x)$$

$$\cos(\varphi + 2\pi) = \cos(\varphi)$$



To pomeni:

$$z = r(\cos(\varphi) + i\sin(\varphi))$$

$$z = r(\cos(\varphi + 2\pi) + i\sin(\varphi + 2\pi))$$

$$z = r(\cos(\varphi + 2k\pi) + i\sin(\varphi + 2k\pi))$$

$$k \in \mathbb{N}$$

$$k \in \mathbb{Z}.$$

Kaj se zgodi s korenom, če namesto φ vstavimo $\varphi + 2k\pi$?

$$w_k = \sqrt[n]{r}\left(\cos\left(\frac{\varphi + 2k\pi}{n}\right) + i\sin\left(\frac{\varphi + 2k\pi}{n}\right)\right)$$

Takoj opazimo: za vsak k velja:

$$w_k^{\,n} = z$$

Res

$$(w_k)^n = \left(\sqrt[n]{r}\right)^n\left(\cos\left(\frac{\varphi+2k\pi}{n}\cdot n\right) + i\sin\left(\frac{\varphi+2k\pi}{n}\cdot n\right)\right)$$

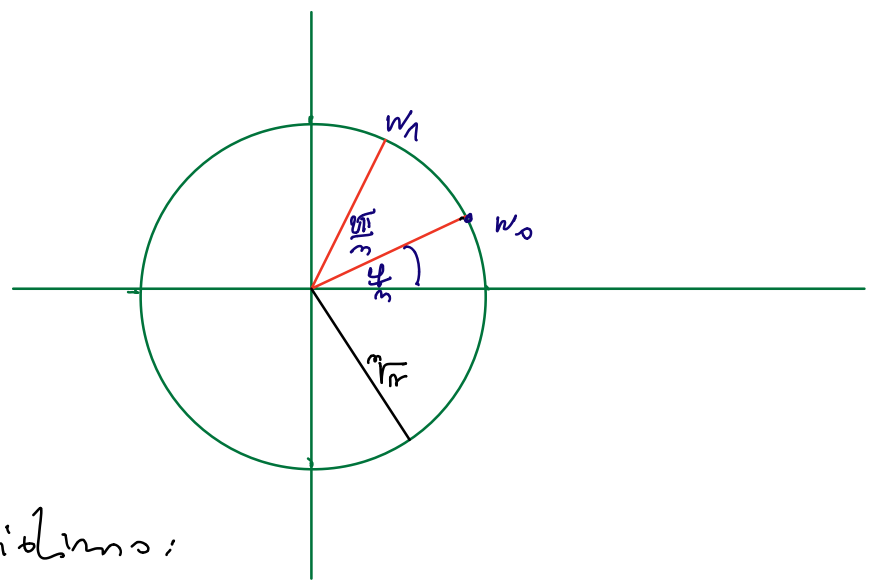
$$= r\left(\cos(\varphi+2k\pi) + i\sin(\varphi+2k\pi)\right) = z$$

Velja pa tudi:

$$w_0 \neq w_1$$

$$w_0 = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n}\right) + i\sin\left(\frac{\varphi}{n}\right)\right)$$

$$w_1 = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + \frac{2\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2\pi}{n}\right)\right)$$



Iščemo rešitve:

$$w_0 = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n}\right) + i\sin\left(\frac{\varphi}{n}\right)\right)$$

$$w_1 = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + \frac{2\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2\pi}{n}\right)\right)$$

$$\vdots$$

$$w_k = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right)\right)$$

$$\vdots$$

$\vdots$

$$w_{n-1} = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + \frac{2\pi(n-1)}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2\pi(n-1)}{n}\right)\right)$$

$$w_n = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + \frac{2\pi n}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2\pi n}{n}\right)\right) =$$

$$= \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + 2\pi\right) + i\sin\left(\frac{\varphi}{n} + 2\pi\right)\right) =$$

$$= \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n}\right) + i\sin\left(\frac{\varphi}{n}\right)\right) =$$

$$= w_0$$

Podobno:

$$w_{n+1} = w_1$$

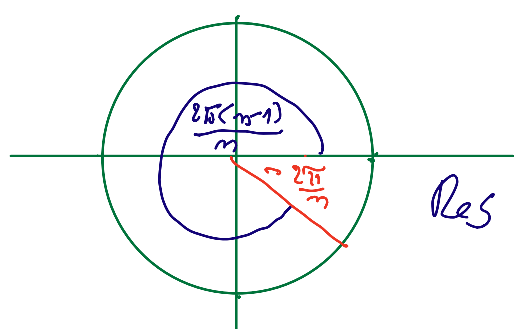
$$w_{n+2} = w_2$$

$\vdots$

Opazimo tudi:

$$w_{-1} = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} - \frac{2\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} - \frac{2\pi}{n}\right)\right)$$

Torej:



$$\frac{2\pi n}{n} = 2\pi = \frac{2\pi(n-1)}{n} + \frac{2\pi}{n}$$

$$\begin{array}{c} \parallel \\ 0 \end{array}$$

$$\frac{2\pi(n-1)}{n} + \frac{2\pi}{n} = 0$$

$$-\frac{2\pi}{n} = \frac{2\pi(n-1)}{n}$$

$$w_{-1} = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + \frac{2\pi(n-1)}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2\pi(n-1)}{n}\right)\right)$$

$$= w_{n-1}$$

Povzemimo:

Koreni

$$w_k = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right)\right)$$

so različni natanko za

$$k = 0, 1, 2, \ldots, n-1$$

Pri drugih $\tilde{k} \in \mathbb{Z}$ velja:

$w_{\tilde{k}} = w_k$ za eno od števil $0, 1, \ldots, n-1$

Velja: Za poljuben $\tilde{k} \in \mathbb{N}$ je:

$$w_{\tilde{k}} = w_k ,$$

kjer je k ostanek pri deljenju $\tilde{k}$ s številom n

Torej: vsako kompleksno število ima natanko n različnih korenov:

Dokazali smo naslednji izrek:

<u>Izrek</u> Naj bo $z \in \mathbb{C}$ poljubno kompleksno število in naj bo $n \in \mathbb{N}$. Število z ima natanko n različnih korenov:

$$z = r(\cos(\varphi) + i\sin(\varphi)),$$

$$w_k = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right)\right)$$

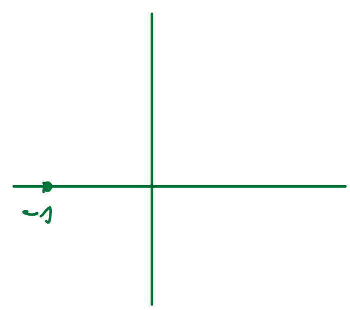
$$k = 0, 1, 2, \ldots, n-1.$$

Koreni $\{w_k \; ; \; k = 0, 1, \dots, n-1\}$ so oglišča pravilnega mnogokotnika D_z^n, ki je včrtan krogu s središčem v izhodišču in z radijem $\sqrt[n]{r}$. Ta mnogokotnik je glede na realno os zasukan za kot $\frac{\varphi}{n}$ v pozitivno smer.

Primeri:

$$z = -1 \; ; \quad \sqrt[4]{z}.$$

$$r = 1 \; ; \quad \varphi = \pi$$



$$-1 = 1(\cos(\pi) + i\sin(\pi))$$

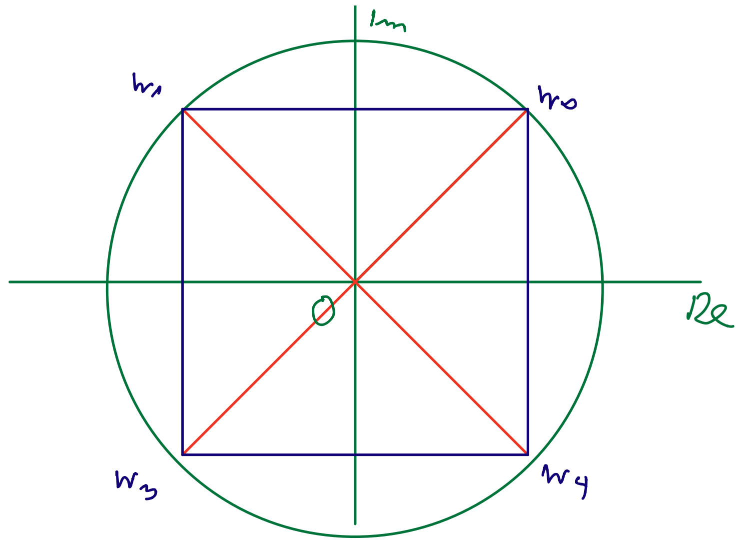
$$w_k = \sqrt[4]{1}\left(\cos\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)\right)$$

$$w_0 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$w_1 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$$

$$w_2 = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)$$

$$w_3 = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)$$



Deljenje kompleksnih števil v polarnih koordinatah.

z, w, iščemo $\frac{z}{w}$

Spomnimo se:

$$\frac{z}{w} = \frac{z \cdot \overline{w}}{w \cdot \overline{w}} = \frac{z \cdot \overline{w}}{|w|^2}$$

$$z = r(\cos(\varphi) + i \sin(\varphi))$$

$$w = s(\cos(\psi) + i \sin(\psi))$$

Poiščimo najprej w^{-1}:

$(w^{-1}) \cdot w = 1$

$$w^{-1} = \frac{1}{w} = \frac{\bar{w}}{w \cdot \bar{w}} = \frac{\bar{w}}{|w|^2}$$

$$w = s\,(\cos(\psi) + i\sin(\psi))$$

$$w^{-1} = \frac{1}{s^2}\big(\, s\,(\cos(\psi) - i\sin(\psi))\,\big) =$$

$$= \frac{1}{s}\,(\cos(-\psi) + i\sin(-\psi))$$

Torej:

$$\boxed{w^{-1} = \frac{1}{s}\,(\cos(-\psi) + i\sin(-\psi))}$$

Upoštevali smo

sodost cos; $\cos(-x) = \cos(x)$

lihost sin; $\sin(-x) = -\sin(x)$

Vrnimo se k $\frac{z}{w}$

$$\frac{z}{w} = z \cdot w^{-1} =$$

$$= r(\cos(\varphi) + i\sin(\varphi)) \cdot \frac{1}{s}(\cos(-\psi) + i\sin(-\psi))$$

(*)

$$= \frac{r}{s}\,(\cos(\varphi - \psi) + i\sin(\varphi - \psi))$$

(*) po našem pravilu za produkt.

Povzemimo:

$$\boxed{\frac{z}{w} = \frac{r}{s}\left(\cos(\varphi-\psi) + i\sin(\varphi-\psi)\right)}$$

$$\boxed{\frac{1}{w} = \frac{1}{s}\left(\cos(-\psi) + i\sin(-\psi)\right)}$$

<u>Množica $\{z \in \mathbb{C} ; |z-\alpha| < r\}$</u>

Naj bo $\alpha = a + ib \in \mathbb{C}$ poljubno število in naj bo $r \in \mathbb{R}^+$ poljubno pozitivno realno število. Poiščimo vsa števila $z = x + iy \in \mathbb{C}$, za katera velja

$$|z - \alpha| < r$$

Oglejmo si najprej

$$|z-\alpha| = r$$

Po definiciji absolutne vrednosti imamo:

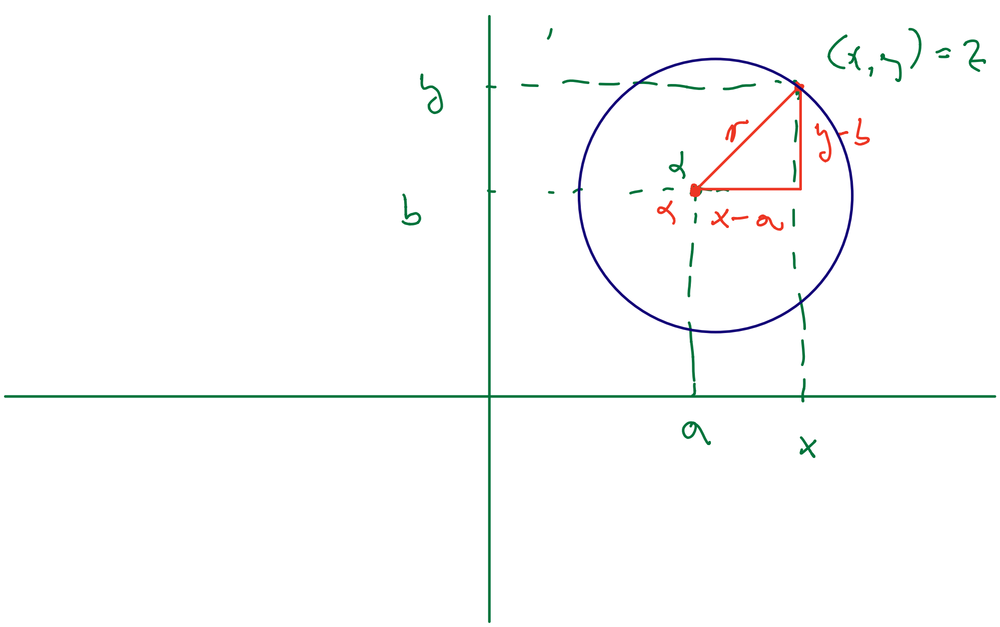
$$|z-\alpha| = |(x+iy)-(a+ib)| =$$

$$= |(x-a)+i(y-b)| =$$

$$\sqrt{(x-a)^2+(y-b)^2}$$

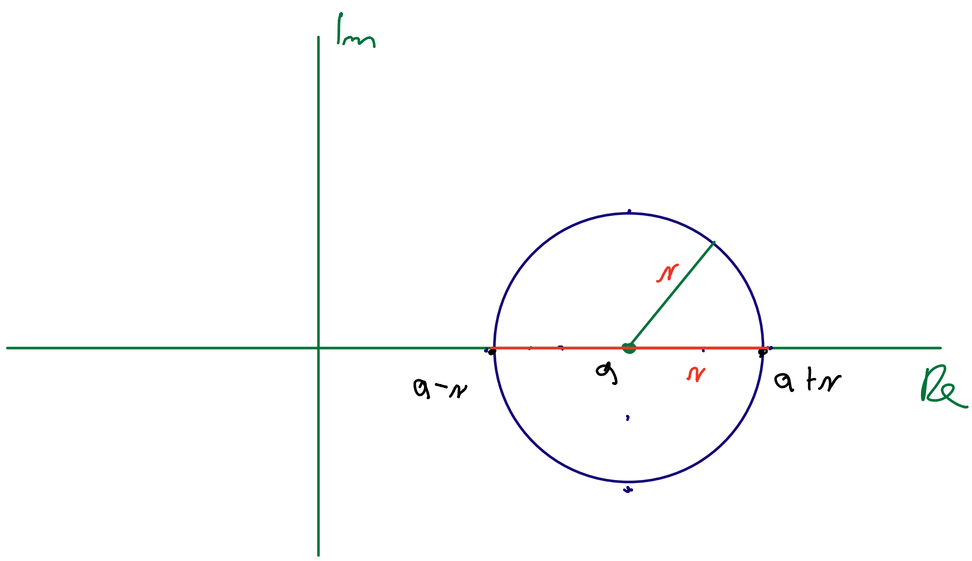
$$\sqrt{(x-a)^2+(y-b)^2} = r$$

$$(x-a)^2+(y-b)^2 = r^2$$



Množica $\{z;\ |z-\alpha| = r\}$ je krožnica s središčem α in radijem r.

Množica $\{z;\ |z-\alpha| < r\}$ je notranjost kroga s središčem α in radijem r.



Matematična analiza

Uvodni razdelek: Preslikave

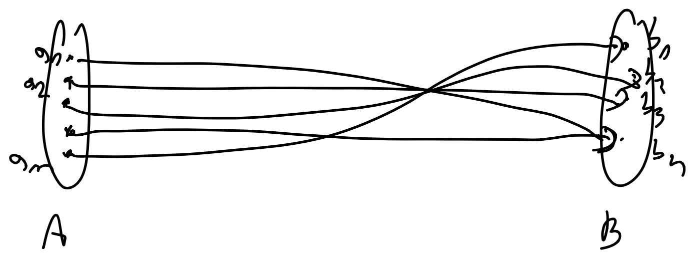
Definicija Naj bosta A in B množici.
Preslikava
$$f: A \longrightarrow B$$
je predpis, ki vsakemu elementu A priredi natanko določen (samo en) element v množici B,
$$f: a \longmapsto b = f(a)$$
original elementa b slika elementa a

A – domena preslikave f
definicijsko območje f

B – zaloga vrednosti f.

Definicija: Preslikava f je surjektivna, če za vsak $b \in B$ obstaja vsaj en $a \in A$ tako, da je

$$f(a) = b$$



Definicija: Preslikava

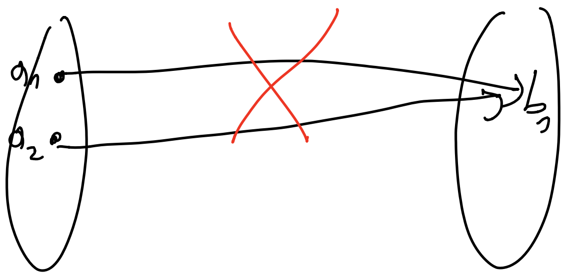
$$f: A \longrightarrow B$$

je injektivna, če velja:

$$f(a_1) = f(a_2) \implies a_1 = a_2$$

oz. ekvivalentno:

$$a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$



To ni injektivnost.

<u>Def:</u> Preslikava

$$f: A \longrightarrow B,$$

ki je hkrati surjektivna in injektivna je bijektivna.