Dokaz $\det P_{ij}A = -\det A$

· Indukcijski korak za 3x3 matrike

$$\det \underbrace{\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{P_{12}A} = (\text{razvijemo po tretji vrstici})$$

$$= a_{31} \det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} \end{bmatrix} \quad a_{31}\left(-\det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}\right.$$

$$- a_{32} \det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} \end{bmatrix} \quad -a_{32}\left(-\det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}\right.$$

$$+ a_{33} \det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} \end{bmatrix} \quad a_{33}\left(-\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right.$$

$$= -\left[a_{31} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} - a_{32} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} + a_{33} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right.$$

$$= (\text{razvoj po tretji vrstici})$$

$$= -\left[\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right]$$

Dokaz $\det A^T = \det A$

$$P_{i,j} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & \cdots & 1 & \\ & & & \vdots & 1 & \vdots & \\ & & & & \ddots & & \\ & & & 1 & \cdots & 0 & \\ & & & & & & 1 \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{bmatrix}\begin{array}{l} \\ \\ \\ i \\ \\ \\ j \\ \\ \\ \\ \end{array} \Rightarrow P_{i,j}^T = \cancel{P} P_{i,j}$$

$$\begin{array}{ccc} & i & j \end{array}$$

$$E_i(z) = \begin{bmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & 1 & & \\ & & & z & \\ & 0 & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ i \\ \\ \\ \\ \end{array} \Rightarrow E_i(z)^T = E_i(z)$$

$$\begin{array}{c} i \end{array}$$

$$E_{i,j}(\alpha) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & \cdots & \alpha & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \begin{array}{l} \\ \\ i \\ \\ j \\ \\ \\ \end{array}$$

$$\begin{array}{cc} i & j \end{array}$$

$$E_{i,j}(\alpha)^T = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & \alpha & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} = E_{j,i}(\alpha)$$

$$\left.\begin{array}{l} \det E_{i,j}(\alpha)^T = \det E_{j,i}(\alpha) = 1 \\ \det E_{i,j}(\alpha) = 1 \end{array}\right\} \begin{array}{l} \det E_{i,j}(\alpha)^T \\ = \det E_{i,j}(\alpha) \end{array}$$

Dokaz $\det\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} = \det\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_{1,3}(b_{11})} \begin{bmatrix} a_{11} & a_{12} & b_{11} & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_{2,3}(b_{21})}$$

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} & b_{11} & 0 \\ a_{21} & a_{22} & b_{21} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_{1,4}(b_{12})} \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{E_{2,4}(b_{22})} \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\parallel$$

$$\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} = E_{2,4}(b_{22})\,E_{14}(b_{12})\,E_{23}(b_{21})\,E_{13}(b_{11})\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

$$\det\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} = \det\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$