$\arcsin(\sin(x))$

$\sin(\arcsin(x))$

Kaj je def. območje? Graf funkcije?

$\sin(\arcsin(x))$

↳ def. območje je $[-1, 1]$

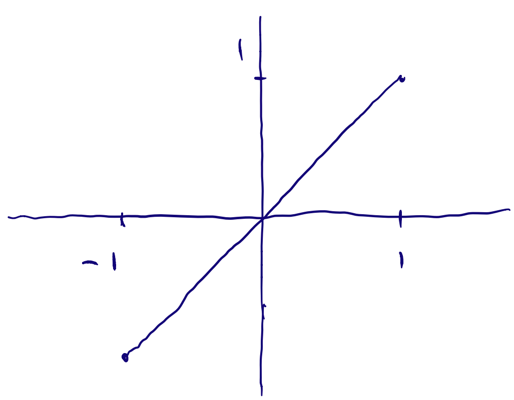
def. območje je $[-1, 1]$

$x \in [-1, 1]$:

$\arcsin x = a \iff \sin a = x$

$\sin(\arcsin(x)) = x$

↳ $x \in [-1, 1]$



$\arcsin(\sin(x))$

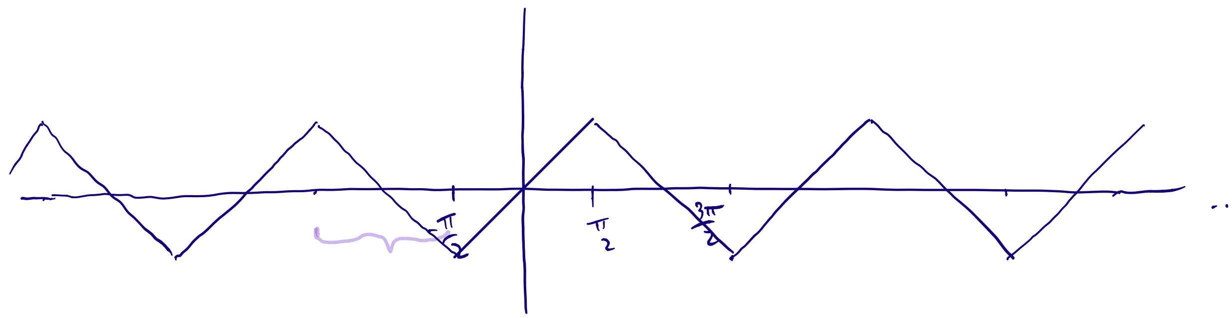
↳ def. območje: $\mathbb{R}$

def. območje: $\mathbb{R}$

$x \in [-1, 1]$; $a \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arcsin x = a \iff \sin a = x$

$x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$: $\arcsin(\sin(x)) = x$



$\sin x = \sin(x + 2k\pi)$ za vsak $k \in \mathbb{Z}$

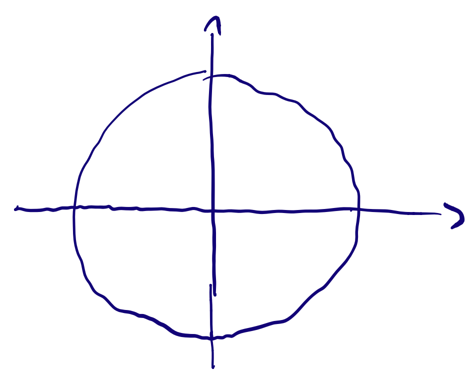
$\arcsin(\ldots)$

$\arcsin(\sin x) = \arcsin(\sin(x + 2k\pi))$; $k \in \mathbb{Z}$

$x \in [\frac{\pi}{2}, \pi]$ $\quad$ $x \in [\frac{\pi}{2}, \frac{\pi}{2}]$

$\arcsin(\sin x) = x$

$[\frac{\pi}{2}, \pi] \rightsquigarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\sin x = -\sin(x + \pi)$

arcsin je liha funkcija

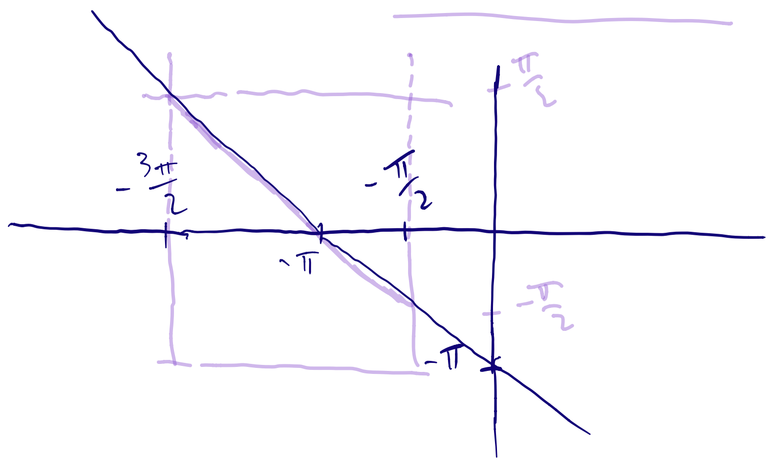
$\arcsin(\sin(x)) = \arcsin(-\sin(x+\pi)) =$

$\hookrightarrow -\sin(x+\pi)$

$$= - \arcsin(\sin(x+\pi)) = -(x+\pi) = -\pi - x$$

$$x \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$$

$$\pi + x \overset{\pi}{\in} \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$a \in \mathbb{R} \setminus \mathbb{Q}$; $f: \mathbb{N} \longrightarrow \mathbb{R}$,

$f(n) = a \cdot n - \lfloor a \cdot n \rfloor$

↳ celi del števila

- f je injektivna
- $Z_f \subseteq (0,1) \cap \mathbb{R} \setminus \mathbb{Q}$

Injektivnost f

$n \neq m, \quad n, m \in \mathbb{N}: \quad f(n) \neq f(m)$

S protislovjem:

$n, m \in \mathbb{N}: \quad f(n) = f(m)$

$$a \cdot n - \underbrace{\lfloor a \cdot n \rfloor}_{\mathbb{Z}} = am - \underbrace{\lfloor am \rfloor}_{\mathbb{Z}}$$

$$\underbrace{\overset{\in \mathbb{R}\setminus\mathbb{Q}}{a}\,(\underbrace{n-m}_{\in \mathbb{Z}})}_{\in \mathbb{R}\setminus\mathbb{Q}} = an - am = \underbrace{\lfloor an \rfloor - \lfloor am \rfloor}_{\in \mathbb{Z} \subseteq \mathbb{Q}}$$

$\cancel{\quad}$

Torej : $f(n) \neq f(m)$, $n \neq m$

· Zaloga vrednosti $f \subseteq [0,1) \cap \mathbb{R}\setminus\mathbb{Q}$

$$f(n) = an - \lfloor an \rfloor$$

$$\forall x \in \mathbb{R} : \quad x - \lfloor x \rfloor \in [0, 1)$$

$$an - \lfloor an \rfloor \neq 0$$

$$\underset{\mathbb{R}\setminus\mathbb{Q}}{an} - \underset{\mathbb{Q}}{\lfloor an \rfloor} \notin \mathbb{Q}$$

Dokaži, da je vsaka monotona $\overset{\text{soda}}{\forall}$ funkcija $f : \mathbb{R} \longrightarrow \mathbb{R}$ konstantna

sodost : $\forall x \in \mathbb{R} : \quad f(x) = f(-x)$

· f naraščajoča (za padajoče tudi podobno)

za vsak $x \in \mathbb{R}$: $f(x) = f(-x)$,

$x_1 < x_2$: $f(x_1) \leq f(x_2)$

$\quad\ \ \shortparallel \qquad\qquad \shortparallel$

$f(x_1)$

$-x_1 > -x_2 : \quad f(-x_1) \geq f(-x_2)$

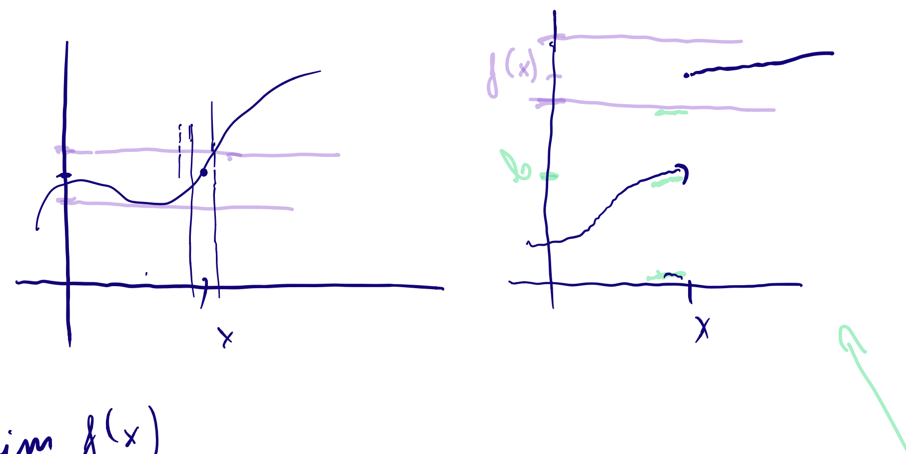
$\hookrightarrow f(x_1) \geq f(x_2)$ $\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow f$ konstanta

ZVEZNE FUNKCIJE

f je zvezna v točki $x \in \mathbb{R}$:

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x' \in \mathbb{R} : |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$



$$\lim_{x \to a} f(x)$$

f je zvezna v $a \in \mathbb{R}$, če je $\lim_{x \to a} f(x) = f(a)$

$$f(x) = \begin{cases} \frac{x^2-1}{x+1} & ; x \neq -1 \\ k & ; x = -1 \end{cases}$$

Najdi $k \in \mathbb{R}$, da bo f zvezna

$$\lim_{x \to -1} \frac{x^2-1}{x+1} = \lim_{x \to -1} \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}} = -2 = f(-1) = k$$

da bo f zvezna

Ls mi smo v okolici -1,
tj. x je v okolici -1, mi pa enak -1

$$\lim_{x \to a} f(x) = b \iff \text{za vsako zaporedje } (a_n)_n \text{ z limito } a \text{ velja: } \lim_{n \to \infty} f(a_n) = b$$

f, g zvezni funkciji, za kateri velja
$f(q) = g(q)$ za vse $q \in \mathbb{Q}$ ✓ (predp.)
$\underline{f(x) = g(x) \text{ za vse } x \in \mathbb{R}}$

$x \in \mathbb{R}$; obstaja zaporedje $(q_n)_n \subseteq \mathbb{Q}$, da je

$x = \lim_{n \to \infty} q_n$

$$f(x) = \lim_{n \to \infty} f(q_n) = \lim_{n \to \infty} g(q_n) = g(x)$$

nekonstantna

Dokaži, da obstaja natanko ena zvezna $f : \mathbb{R} \to \mathbb{R}$, za katero velja:

- $f(x+y) = f(x) + f(y)$
- $\underline{f(xy) = f(x)f(y)}$

$\Bigg\}\ *$

Naj za zvezno f velja $*$.
Dokazujemo: $f = id_{\mathbb{R}}$

$f = id_{\mathbb{R}}$ ✓

$f(n) = n$, za vse $n \in \mathbb{N}$

$f(1) = 1$ (dobimo iz *)

$1 = 1 \cdot 1$

$f(1) = f(1)^2 \Rightarrow f(1) = 0$ ali $f(1) = 1$

$x = 1 \cdot x$ za vse $x \in \mathbb{R}$

$f(x) = f(1 \cdot x) = 0 \cdot f(x) \Rightarrow f(x) = 0$

$f(n) = n$ (I.P.)

$f(n+1) = \underset{\text{I.P.}}{f(n)} + 1 = n+1$

$f(m) = m$ za $m \in \mathbb{Z}$

$f(-1) = ?$

$f(0) = 0$ (premisli)

$f(-1)$

$f(1-1) = f(1 + (-1)) = \overset{1}{f(1)} + f(-1)$

$\Rightarrow f(-1) = -1$

isti argument uporabimo za vsak $n \in \mathbb{N}$

(namesto 1 pišemo: $0 = n + (-n)$ ....)

$f\left(\frac{n}{m}\right) = f(n) f\left(\frac{1}{m}\right) = n \cdot \frac{1}{m} = \frac{n}{m}$

homogenost

$f\left(\frac{1}{n}\right) = \frac{1}{n}$ za $\forall n > 0$?

$1 = n \cdot \frac{1}{n}$

$\overset{\shortparallel}{f(1)} = \underset{\overset{\shortparallel}{n}}{f(n)} \cdot f\left(\frac{1}{n}\right) \Rightarrow f\left(\frac{1}{n}\right) = \frac{1}{n}$ ✓

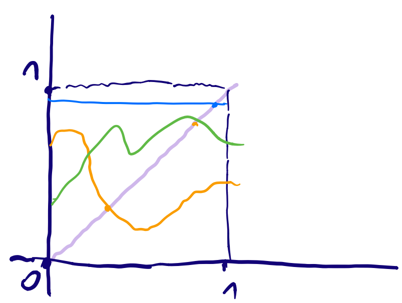
Zdaj smo pokazali: $f(q) = q$ za vse $q \in \mathbb{Q}$

$\Longrightarrow f(x) = x$ za vse $x \in \mathbb{R}$

f je zvezna

$f : [0,1] \longrightarrow [0,1]$; zvezna

$\exists x \in [0,1] : f(x) = x$



Namig: oglejmo si funkcijo

$g(x) := f(x) - x$

x je fiksna točka za f $(f(x) = x)$

$\Leftrightarrow g(x) = 0$

Pokažimo, da ima g ničlo!

$g(1) \leq 0$

$\overset{\shortparallel}{f(1)} - 1 \leq 1 - 1 = 0$

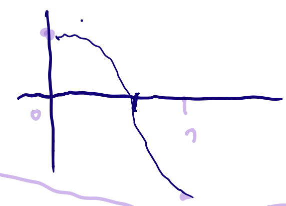
$\hookrightarrow f(1) \leq 1$

$g(0) \geq 0$

(isti arg.)

g je zvezna

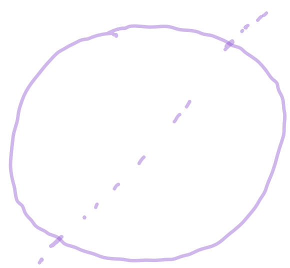
- če $g(1) = 0$ ✓
- če $g(0) = 0$ ✓
- če je $g(1) < 0$ in $g(0) > 0$



Sledi: g ima med 0 in 1 ničlo!

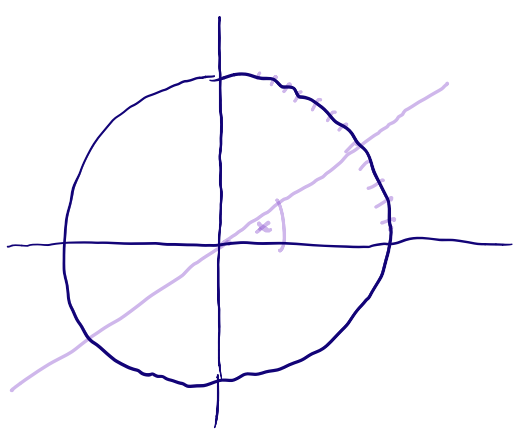
$f(x) - x = g(x) = 0$ za $x \in (0,1)$

$f(x) = x$ za neki $x \in [0,1]$



Dokaži, da obstajata dve diam. nasprotni točki na ekvatorju, kjer je temperatura enaka.

(temperatura je zvezna funkcij



gledamo $T(x)$, kjer je x kot

$T(x) = T(x + 2\pi)$ za vse x

Hočemo: $x \in \mathbb{R}: T(x) = T(x + \pi)$

$V(x) := T(x) - T(x + \pi)$

Iščemo ničle ✓

- T zvezna $\Rightarrow$ V zvezna
- T period. s periodo 2π $\Rightarrow$ V period. s periodo 2π

• Ločimo tri primere:

1. $V(0) = 0$ ✓
2. $V(0) > 0 \iff T(0) > T(\pi)$
3. $V(0) < 0$

$V(0) > 0 \implies V(\pi) < 0$

$$V(\pi) = T(\pi) - \underbrace{T(2\pi)}_{=T(0)} = -V(0)$$

$V(0)V(\pi) < 0 \implies \exists x \in (0,\pi): V(x) = 0$ oz. $T(x) = T(x+\pi)$ ✓

$V(0) < 0$ ... analogno ✓

Obstaja $x \in \mathbb{R}$, $V(x) = 0$ oz. $T(x) = T(x+\pi)$

f zvezna na $[x_1, x_2]$, potem f zavzame vse vrednosti med $f(x_1)$ in $f(x_2)$ na (x_1, x_2)

$$g(x) := f(x) - a$$

Iščemo ničlo od g na (x_1, x_2), $g(x_1)g(x_2) < 0$

$\implies$ ničla g obstaja

$a \in (f(x_1), f(x_2))$ (če je $f(x_1) < f(x_2)$, sicer $x_1 \xleftrightarrow{\text{zamenjamo}} x_2$)

NAMIG ZA 10: indukcija po n

f, g funkciji: $\quad \cdot \lim_{x \to a} f(x) = \alpha > 0$

$a, \cancel{b} \in \mathbb{R}$ $\quad \cdot \lim_{x \to b} g(x) = \beta \in \mathbb{R}$

$$\lim_{x \to a} f(x)^{g(x)} = \alpha^{\beta}$$

Namig: $\quad a^b, \; a > 0: \quad e^{\ln(a^b)} = a^b$

$$e^{\ln(a^b)} = e^{\underline{b \ln(a)}}$$

$$\lim_{x \to a} \underbrace{f(x)^{g(x)}} = e^{\beta \ln \alpha} = \alpha^{\beta}$$

$$f(x)^{g(x)} = e^{g(x) \cdot \ln(f(x))}$$

$g(x) \xrightarrow{x \to a} \beta \qquad \ln(f(x)) \xrightarrow{x \to a} \ln(\alpha)$

Izračunaj limite

$$\lim_{x \to 0} x \underbrace{\sin\left(\frac{1}{x}\right)} = 0$$

$x \xrightarrow{x \to 0} 0$

$\hookrightarrow -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$$\lim_{x \to 0} \frac{\sin(x)}{|x|}$$

$x < 0; \quad |x| = -x$

$$\hookrightarrow \lim_{x \uparrow 0} \frac{\sin(x)}{|x|} = \lim_{x \uparrow 0} \frac{\sin(x)}{-x} = -1$$

$$\hookrightarrow \lim_{x \downarrow 0} \frac{\sin(x)}{|x|} = \lim_{x \downarrow 0} \frac{\sin(x)}{x} = 1$$

$x > 0$

$|x| = x$

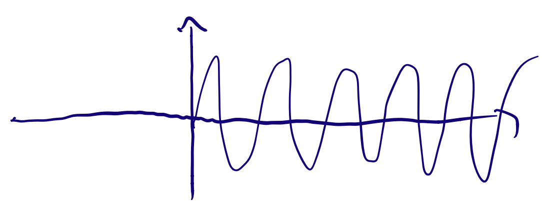
$$\lim_{x \to \infty} \frac{x : x}{\sqrt{x^2+2} : x} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + \underset{\downarrow 0}{\frac{2}{x^2}}}} = 1$$

$$\lim_{\substack{x \to \infty \\ x \in \mathbb{R}}} f(x) = a \Rightarrow \lim_{n \to \infty}$$

$$\lim_{\substack{x \to \infty \\ x \in \mathbb{R}}} f(x) = a \Rightarrow \lim_{\substack{n \to \infty \\ n \in \mathbb{N}}} f(n) = a$$

$$\overset{?}{\not\Leftarrow}$$

$$\lim_{n \to \infty} \underbrace{\sin(2n\pi)}_{0} = 0 \not\Rightarrow \lim_{x \to \infty} \sin(2\pi x) = 0$$



Vemos: $\left.\begin{array}{l} \lim\limits_{\substack{x \to \infty \\ x \in \mathbb{R}}} f(x) = \overset{?}{a} \in \mathbb{R} \\ \\ \lim f(n) = b \end{array}\right\} a = b$

$n \to \infty$
$n \in \mathbb{N}$

$\lim_{x \to 0} \left(1 + e^{\frac{1}{x}}\right)^{-1}$ ... ne obstaja

$\lim_{x \uparrow 0} \left(1 + e^{\frac{1}{x}}\right)^{-1} = 1$

$\lim_{x \downarrow 0} \left(1 + \underbrace{e^{\frac{1}{x}}}_{\infty}\right)^{-1} = 0$

$\frac{\sin x}{x} \xrightarrow{x \to 0} 1$

$\lim_{x \to 0} \left(\underbrace{1 + 3x}_{1}\right)^{\frac{2}{\sin x}} = *$

$\frac{2}{\sin x} \to \infty$

1. način: $(1 + 3x)^{\frac{2}{\sin x}} = e^{\frac{2}{\sin x} \ln(1+3x)}$ ↳ lim

$* = \lim_{x \to 0} (1 + 3x)^{\frac{1}{x} \cdot x \cdot \frac{2}{\sin x}} =$

$= \lim_{x \to 0} \left[(1 + 3x)^{\frac{1}{x}}\right]^{\underbrace{\frac{x}{\sin x} \cdot 2}_{2}} = e^6$

$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

$(1 + 3x)^{\frac{1}{x}}$

$\lim_{x \to 0} (1 + 3x)^{\frac{1}{x}} = e^3$

D: $\lim\limits_{x \downarrow 0} (1+3x)^{\frac{1}{x}} = \lim\limits_{t \to \infty} \left(1+\frac{3}{t}\right)^{t} = \lim\limits_{t \to \infty} \left[\left(1+\frac{1}{\frac{t}{3}}\right)^{\frac{t}{3}}\right] = e^{3}$

$t := \frac{1}{x}$

α: $\lim\limits_{x \uparrow 0} (1+3x)^{\frac{1}{x}} = e^{3}$

$\lim\limits_{x \to 1} x^{\tan\left(\frac{\pi x}{2}\right)} =$

$x^{\tan\left(\frac{\pi x}{2}\right)} = e^{\underbrace{\tan\left(\frac{\pi x}{2}\right) \ln x}}$

$\lim\limits_{x \to 1} \underbrace{\tan\left(\frac{\pi x}{2}\right)}_{\to \infty} \underbrace{\ln x}_{\to 0} = \lim\limits_{x \to 1} \sin\left(\frac{\pi x}{2}\right) \cdot \frac{1}{\cos\left(\frac{\pi x}{2}\right)} \ln x$

NAMIG: $\cos\left(\frac{\pi x}{2}\right) \ldots \sin(\ldots)$