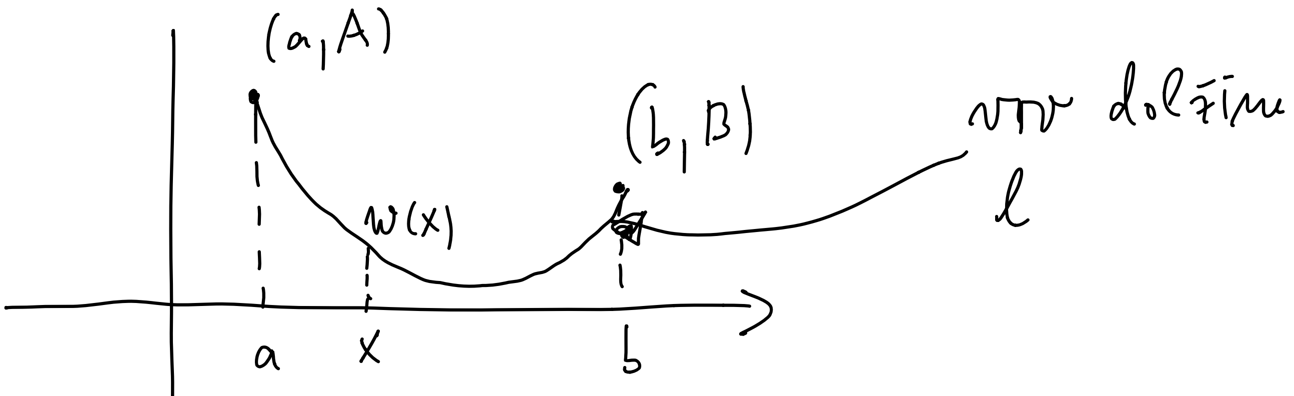
Zvezna verižnica (nadaljevanje)

$$w(x) - \lambda = C\sqrt{1 + w'^2(x)}$$



Uporabimo nastavek:

$$w' = \sinh p \ , \quad p \text{ neka funkcija } x$$

Potem dobimo

$$w = \lambda + C\cosh p \quad / \text{ diferenciramo}$$

$$dw = C \sinh p \, dp$$

$$w' dx = C \sinh p \, dp$$

$$\sinh p \, dx = C \sinh p \, dp \quad / : \sinh p$$

$$dx = C \, dp \quad / \int$$

$$\boxed{x = Cp + D} \Rightarrow p(x) = \frac{x-D}{C}$$

Torej je

$$\boxed{w(x) = \lambda + \cosh \frac{x-D}{C}}$$

λ, C in D neznane konstante!!

Upoštevamo dolžino vrvi:

$$\ell = \int_a^b \sqrt{1 + w'(x)^2}\, dx$$

$$= \int_a^b \cosh\left(\frac{x-D}{C}\right) dx$$

$$= C \sinh\left(\frac{x-D}{C}\right)\Big|_a^b$$

$$(*) \qquad = \boxed{C \sinh \frac{b-D}{C} - C \sinh \frac{a-D}{C} = \ell}$$

Ideja: še dve enačbi za λ, C in D!

Upoštevati moramo še, da je vv na

robnih pripeta v (a, A) in (b, B):

$$w(a) = A = \lambda + C\cosh\left(\frac{a-D}{C}\right) \quad (**)$$

$$w(b) = B = \lambda + C\cosh\left(\frac{b-D}{C}\right) \quad (***)$$

Uvedemo novi spremenljivki

$$\mu = \frac{a-D}{C} \quad \text{in} \quad \nu = \frac{b-D}{C}.$$

Dobimo poenostavljen sistem:

$$\boxed{\begin{aligned} \frac{A}{C} &= \frac{\lambda}{C} + \cosh\mu \\ \frac{B}{C} &= \frac{\lambda}{C} + \cosh\nu \\ \frac{\ell}{C} &= \sinh\nu - \sinh\mu \end{aligned}}$$

Ta sistem prevedemo v sistem

$$\frac{B}{C} - \frac{A}{C} = \cosh\nu - \cosh\mu = \frac{B-A}{C}$$

$$\Rightarrow \boxed{\frac{B-A}{b-a} = \frac{\cosh\nu - \cosh\mu}{\nu-\mu}}$$

$$\boxed{\frac{l}{b-a} = \frac{\sinh v - \sinh u}{v-u}}$$

Uporabimo adicijske izreke za hiperbolične funkcije:

$$(\Delta) \quad \frac{B-A}{b-a} = \frac{\sinh\frac{u+v}{2}\,\sinh\frac{v-u}{2}}{\frac{v-u}{2}}$$

$$\frac{l}{b-a} = \frac{\cosh\frac{v+u}{2}\,\sinh\frac{v-u}{2}}{\frac{v-u}{2}}$$

$$\Rightarrow \boxed{\frac{B-A}{l} = \tanh\frac{u+v}{2}}$$

$$\Rightarrow \frac{u+v}{2} = \operatorname{atanh}\frac{B-A}{l} \quad / \sinh$$

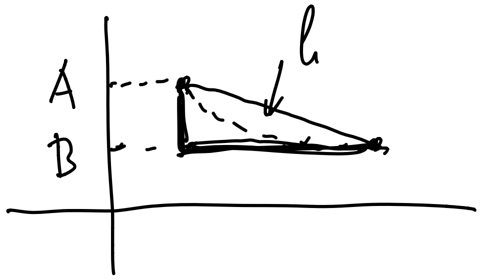
$$\sinh\frac{v+u}{2} = \sinh\left(\operatorname{atanh}\frac{B-A}{l}\right)$$

$$= \frac{\tanh\frac{u+v}{2}}{\sqrt{1-\tanh^2\left(\frac{v+u}{2}\right)}}$$

$$= \frac{\frac{B-A}{\ell}}{\sqrt{1-\left(\frac{B-A}{\ell}\right)^2}} = \frac{(B-A)\cancel{\ell}}{\cancel{\ell}\sqrt{\ell^2-(B-A)^2}}$$

Opomba: $\ell^2-(B-A)^2 > 0$?

Ekvivalentno: $\ell > |B-A|$?



$$\ell > h = \sqrt{(b-a)^2+(B-A)^2}$$
$$> \sqrt{(B-A)^2} = |B-A|$$

Če pravkar izpeljano zvezo upoštevamo v enačbi (Δ), dobimo:

$$\sinh\frac{\nu-\mu}{2} = \rho\,\frac{\nu-\mu}{2},$$ kjer je

$$\rho = \frac{\sqrt{\ell^2-(B-A)^2}}{b-a} > 1$$ doma

Rešujemo torej enačbo

$$\boxed{\sinh z = \rho z}, \quad z = \frac{\nu-\mu}{2}.$$

Navadna iteracija:

$$z_{k+1} = g(z_k), \quad g \text{ ustrezna}$$

① $$g(z) = \frac{\sinh z}{\rho}$$

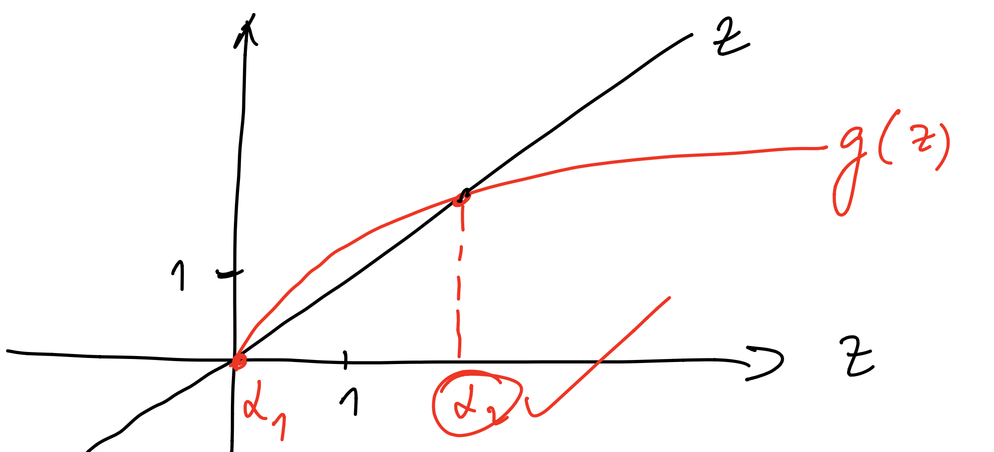
② $$\boxed{g(z) = \operatorname{asinh}(\rho z)}$$

Kdaj bo zaporedje $z_{k+1} = g(z_k)$ konvergiralo k negibni točki α:

$$\alpha = g(\alpha) \quad (\text{rešitev prvotne enačbe})\,?$$

Mora veljati: $|g'(\alpha)| < 1$!!!

Izberimo g iz (2): $g(z) = \operatorname{asinh}(\rho z), \rho > 1.$



Videti moramo, da je $|g'(\alpha_2)| < 1$!

$$g'(z) = (\operatorname{asinh}(\rho z))'$$

$$= \frac{\rho}{\sqrt{1+z^2\rho^2}}$$

$$\Rightarrow g'(0) = g'(\alpha_1) = \rho > 1 \text{ (odbojna)}$$

$$g'(\alpha_2) = \frac{\rho}{\sqrt{1+\alpha_2^2\rho^2}} =$$

$$\boxed{\alpha_2 = \operatorname{asinh} \rho \alpha_2}$$

$$\rho \alpha_2 = \sinh \alpha_2$$

$$\boxed{\rho = \frac{\sinh \alpha_2}{\alpha_2}}$$

doma ali vaj

$$= \dots = \frac{\tanh \alpha_2}{\alpha_2}$$

Želimo pokazati, da je $\frac{\tanh \alpha_2}{\alpha_2} < 1$

Če definiramo $h(\alpha) = \tanh(\alpha)/\alpha \Rightarrow$

① $h(0) = 1$ (limita)

② $h'(\alpha) < 0, \ \alpha > 0$ (doma)

$\Rightarrow h$ padajoča na $(0, \infty) \Rightarrow$

$$\boxed{h(\alpha) < 1 \text{ za } \alpha > 0}$$