PRESLIKAVE

Preslikava $F : \mathbb{R}^n \to \mathbb{R}^m$:

$f_1, \dots, f_m$ so funkcije n spremenljivk

$$F(x_1, x_2, \dots, x_n) = F(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$$

Preslikava F je **diferenciabilna** v $\vec{a} \in D_F$, če obstaja matrika A dimenzije $m \times n$, za katero velja:

$$F(\vec{a} + \vec{h}) = F(\vec{a}) + A \cdot \vec{h} + o(\vec{h}); \qquad \lim_{||\vec{h}|| \to 0} \frac{||o(\vec{h})||}{||\vec{h}||} = 0$$

Diferencial $dF(\vec{a})$ (diferenciabilne) preslikave $\vec{f}$ v točki $\vec{a}$ je linearna preslikava z matriko A:

$$(dF)(\vec{a})(\vec{h}) = A\vec{h}$$

Matriki A pravimo **Jacobijeva matrika** JF:

$$JF(x_1, x_2, \dots, x_n) = \begin{bmatrix} df_1(\vec{x}) \\ df_2(\vec{x}) \\ \vdots \\ df_m(\vec{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

41. Naj bo $F : \mathbb{R}^2 \to \mathbb{R}^3$ podana s predpisom $f(x, y) = ((x + y^2)^3, \sin(x \cdot y^2), \ln(1 + x^2 + y^2))$. Izračunaj $JF(0, 1)$.

$$JF(x,y) = \begin{bmatrix} 3 \cdot (x+y^2)^2 \cdot 1 & 3 \cdot (x+y^2)^2 \cdot 2y \\ \cos(x \cdot y^2) \cdot y^2 & \cos(x \cdot y^2) \cdot 2xy \\ (1+x^2+y^2)^{-1} \cdot 2x & (1+x^2+y^2)^{-1} \cdot 2y \end{bmatrix} \qquad JF(0,1) = \begin{bmatrix} 3 & 6 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

42. Naj bo $F(x) = (x^5, \tan x^2, \arctan x^2)$.

a) Določi definicijsko območje. b) Izračunaj Jacobijevo matriko.

a) $D_{\tan} = \mathbb{R} \setminus \{ \frac{\pi}{2} + k\pi \; ; \; k \in \mathbb{Z} \}$

$x^2 \neq \frac{\pi}{2} + k\pi \; ; \; k \in \mathbb{Z} \qquad x^2 \geq 0 \Rightarrow k \in \mathbb{N}_0$

$x \neq \pm \sqrt{\frac{\pi}{2} + k\pi} \; ; \; k \in \mathbb{N}_0$

$D_F = \mathbb{R} \setminus \{ \pm \sqrt{\frac{\pi}{2} + k\pi} \; ; \; k \in \mathbb{N}_0 \}$

b)

$$JF = \begin{bmatrix} 5x^4 \\ \frac{1}{\cos^2 x^2} \cdot 2x \\ \frac{1}{1+x^4} \cdot 2x \end{bmatrix}$$

MATRIČNO VERIŽNO PRAVILO

F, G diferenciabilni preslikavi

$$d(F \circ G)(\vec{x}) = dF(G(\vec{x}))\, dG(\vec{x})$$

43. Spremenljivki u in v sta diferenciabilni funkciji spremenljivk x, y in z, le-te pa so nadaljnje diferenciabilne funkcije spremenljivk t in s. V neki točki velja:

$$\frac{\partial u}{\partial x} = 2, \qquad \frac{\partial u}{\partial y} = 3, \qquad \frac{\partial u}{\partial z} = 1$$

$$\frac{\partial v}{\partial x} = -1, \qquad \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial v}{\partial z} = 3$$

$$\frac{\partial x}{\partial s} = 5, \qquad \frac{\partial y}{\partial s} = -6, \qquad \frac{\partial z}{\partial s} = 4$$

$$\frac{\partial x}{\partial t} = -3, \qquad \frac{\partial y}{\partial t} = 2, \qquad \frac{\partial z}{\partial t} = 3$$

Spremenljivki u in v lahko gledamo tudi kot funkciji spremenljivk s in t. V dani točki izračunajte parcialne odvode $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial s}$ in $\frac{\partial v}{\partial t}$.

$u = f_1(x, y, z)$ $\qquad$ $x = g_1(t, s)$

$v = f_2(x, y, z)$ $\qquad$ $y = g_2(t, s)$

$\qquad\qquad\qquad\qquad$ $z = g_3(t, s)$

$(u, v) = F(x, y, z)$

$(x, y, z) = G(t, s)$

$$dF = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 3 \end{bmatrix} \qquad dG = \begin{bmatrix} 5 & -3 \\ -6 & 2 \\ 4 & 3 \end{bmatrix}$$

$$d(F \circ G) = dF\, dG = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & -3 \\ -6 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 7 & 12 \end{bmatrix}$$

$$d(F \circ G) = \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}$$

$$\Rightarrow \frac{\partial u}{\partial s} = -4, \quad \frac{\partial u}{\partial t} = 3, \quad \frac{\partial v}{\partial s} = 7, \quad \frac{\partial v}{\partial t} = 12$$

44. Spremenljivke u, v in w so diferenciabilne funkcije spremenljivk x in y, le-ti pa se izražata v polarnih koordinatah: $x = r\cos\varphi$, $y = r\sin\varphi$. Pri $x = 3$ in $y = 4$ velja:

$$\frac{\partial u}{\partial x} = 2, \qquad \frac{\partial u}{\partial y} = -1,$$
$$\frac{\partial v}{\partial x} = 3, \qquad \frac{\partial v}{\partial y} = 0,$$
$$\frac{\partial w}{\partial x} = 1, \qquad \frac{\partial w}{\partial y} = 2.$$

Spremenljivke u, v in w lahko gledamo tudi kot funkcije spremenljivk r in φ. V dani točki izračunajte vse možne parcialne odvode prvega reda spremenljivk u, v in w po spremenljivkah r in φ.

$$u = f_1(x,y,z) \qquad x = g_1(r,\varphi) = r\cdot\cos\varphi$$
$$v = f_2(x,y,z) \qquad y = g_2(r,\varphi) = r\cdot\sin\varphi$$
$$w = f_3(x,y,z)$$

$x=3, y=4$: $\quad dF(3,4) = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} \qquad dG(r,\varphi) = \begin{bmatrix} \cos\varphi & -r\cdot\sin\varphi \\ \sin\varphi & r\cdot\cos\varphi \end{bmatrix}$

$$x=3, y=4 \Rightarrow \left.\begin{array}{l} r\cdot\cos\varphi = 3 \\ r\cdot\sin\varphi = 4 \end{array}\right\} r^2 = 3^2+4^2 = 9+16 = 25 \Rightarrow r = 5$$

$$5\cdot\cos\varphi = 3 \Rightarrow \cos\varphi = \frac{3}{5} \qquad \Rightarrow \varphi = \arccos\left(\frac{3}{5}\right) \quad (\text{ker je } y>0)$$
$$5\cdot\sin\varphi = 4 \Rightarrow \sin\varphi = \frac{4}{5}$$

$$dG\left(5, \arccos\left(\frac{3}{5}\right)\right) = \begin{bmatrix} \frac{3}{5} & -4 \\ \frac{4}{5} & 3 \end{bmatrix}$$

Matrično verižno pravilo:

$$\begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \varphi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \varphi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \varphi} \end{bmatrix} = d(F\circ G) = dF\,dG = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -4 \\ \frac{4}{5} & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -11 \\ \frac{9}{5} & -12 \\ \frac{11}{5} & 2 \end{bmatrix}$$

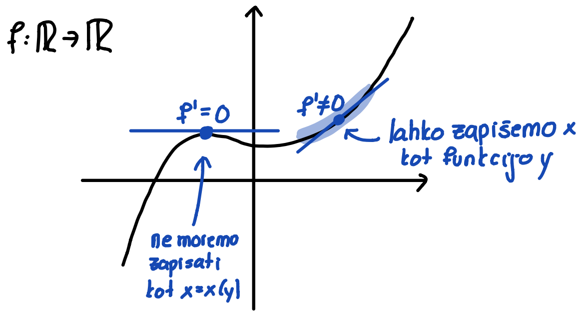
$$\Rightarrow \frac{\partial u}{\partial r} = \frac{2}{5}, \ldots, \frac{\partial w}{\partial \varphi} = 2$$

IZREK O INVERZNI PRESLIKAVI

24.11.2020

Izrek: Naj bo $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ diferenciabilna v $\vec{a}$ in naj bodo vsi odvodi zvezni. Če velja $\det JF(\vec{a}) \neq 0$, obstaja okolica U točke $\vec{a}$ in okolica V točke $F(\vec{a})$, da je $F: U \to V$ obrnljiva, za njen lokalni inverz $G: V \to U$ pa velja:

$$JG(F(\vec{a})) = [JF(\vec{a})]^{-1}$$



45. Dan je sistem enačb:

$$(1+y^2)\ln x = z$$
$$x^3 + xy = w.$$

a) Rešite ta sistem za $z = 0$ in $w = 5$.

b) Pokažite, da obstajata taka okolica U rešitve iz prejšnje točke in taka okolica V točke $(0,5)$, da ima sistem za vsak par $(z,w) \in V$ enolično rešitev na $(x,y) \in U$.

c) Pri $z = 0$ in $w = 5$ izračunajte parcialne odvode $\frac{\partial x}{\partial z}$, $\frac{\partial x}{\partial w}$, $\frac{\partial y}{\partial z}$ in $\frac{\partial y}{\partial w}$

d) Z uporabo diferenciabilnosti približno rešite sistem za z = 0.17 in w = 5.1: kot izhodiščni približek vzemite rešitev iz točke a) in uporabite tamkajšnje parcialne odvode, izračunane v prejšnji točki.

a) $(1+y^2)\cdot \ln x = 0 \quad \to \quad \underline{\underline{x=1}}$, ker je $1+y^2 \neq 0$

$x^3 + xy = 5 \quad \to \quad 1 + y = 5 \to \underline{\underline{y=4}}$

b) $\vec{a} = (1,4)$

$F(\vec{a}) = (0,5)$

$$F(x,y) = \begin{bmatrix} (1+y^2)\ln x \\ x^3 + xy \end{bmatrix}$$

$$\frac{\partial z}{\partial x} = (1+y^2)\cdot\frac{1}{x} \qquad \frac{\partial z}{\partial y} = 2y\cdot\ln x$$

$$\frac{\partial w}{\partial x} = 3x^2+y \qquad \frac{\partial w}{\partial y} = x$$

$\Big\}$ zvezne funkcije, F diferenciabilna

$x=1, y=4$:

$$JF = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 7 & 1 \end{bmatrix}$$

Rešitev sledi iz izreka o inverzni preslikavi.

c) Uporabimo izrek o inverzni preslikavi:

$$J(F^{-1})(0,5) = (JF(1,4))^{-1}$$

$* A \in \mathbb{R}^{2\times 2}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\cdot\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\frac{1}{ad-bc}\cdot\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = I\right)$$

$$\begin{bmatrix} \frac{\partial x}{\partial z}(0,5) & \frac{\partial x}{\partial w}(0,5) \\ \frac{\partial y}{\partial z}(0,5) & \frac{\partial y}{\partial w}(0,5) \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 7 & 1 \end{bmatrix}^{-1} =$$

$$= \frac{1}{17\cdot 1 - 7\cdot 0}\begin{bmatrix} 1 & 0 \\ -7 & 17 \end{bmatrix} = \begin{bmatrix} \frac{1}{17} & 0 \\ -\frac{7}{17} & 1 \end{bmatrix}$$

d) $z \doteq 0{,}17$

$w \doteq 5{,}1$

$$\underbrace{F^{-1}(z,w) = (x,y) \Leftrightarrow F(x,y) = (z,w)}_{\text{oboje predstavlja isti sistem v okolici neke točke}}$$

$$F^{-1}(z,w) \approx F^{-1}(z_0,w_0) + J(F^{-1}(z_0,w_0))\cdot\begin{bmatrix} z-z_0 \\ w-w_0 \end{bmatrix}$$

$$F^{-1}\left(\begin{bmatrix} 0{,}17 \\ 5{,}1 \end{bmatrix}\right) \approx F^{-1}\left(\begin{bmatrix} 0 \\ 5 \end{bmatrix}\right) + \begin{bmatrix} \frac{1}{17} & 0 \\ -\frac{7}{17} & 1 \end{bmatrix}\cdot\begin{bmatrix} 0{,}17-0 \\ 5{,}1-5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} \frac{0{,}17}{17}+0 \\ -7\cdot\frac{0{,}17}{17}+0{,}1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0{,}01 \\ 0{,}03 \end{bmatrix} = \begin{bmatrix} 1{,}01 \\ 4{,}03 \end{bmatrix}$$

$\Rightarrow x \doteq 1{,}01$

$y \doteq 4{,}03$

IZREK O IMPLICITNI PRESLIKAVI

Izrek: Naj bo $F : D \subseteq \mathbb{R}^{n+m} \times \mathbb{R}^m \to \mathbb{R}^m$ zvezno odvedljiva preslikava, $(\vec{a}, \vec{b}) \in D$. Če je

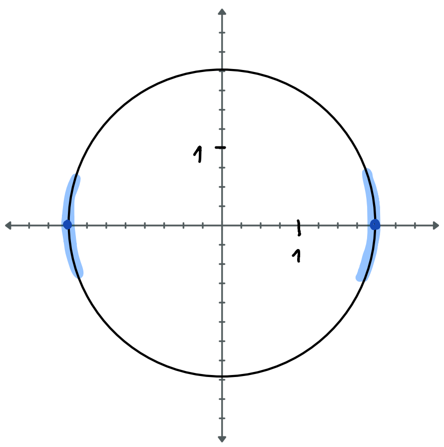
(i) $F(\vec{a}, \vec{b}) = 0$,

(ii) $\det JF^{\vec{y}}(\vec{a}, \vec{b}) \neq 0$,

obstaja okolica U od $\vec{a}$, na kateri je enolično definirana preslikava $\vec{g} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, za katero velja:

- $\vec{g}(\vec{a}) = \vec{b}$
- $F(\vec{x}, \vec{g}(\vec{x})) = 0$, $\vec{x} \in U$
- $J\vec{g}(\vec{a}) = -[JF^{\vec{y}}(\vec{a}, \vec{b})]^{-1} \cdot JF^{\vec{x}}(\vec{a}, \vec{b})$

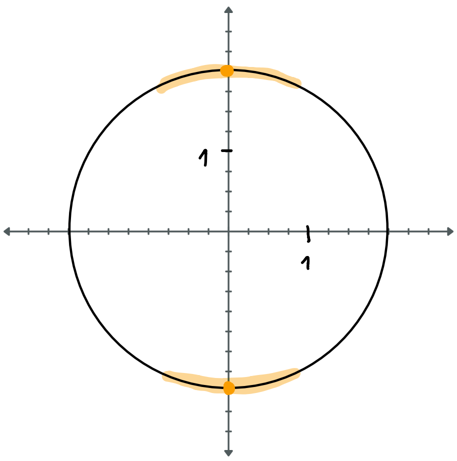
Primer: krožnica z enačbo $x^2+y^2=4$



$F(x,y) = x^2+y^2-4$

V okolici katerih točk lahko del krožnice opišemo kot $y=g(x)$?

- v vseh razen v točkah $T(2,0)$ in $T(-2,0)$



$F(x,y) = x^2+y^2-4$

V okolici katerih točk lahko del krožnice opišemo kot $x=g(y)$?

- v vseh razen v točkah $T(0,2)$ in $T(0,-2)$

Pokaži, da lahko rešitev enačbe $z\cdot e^{xy}+e^z=0$ v okolici točke $(1,-1,-1)$ zapišemo kot funkcijo $z=z(x,y)$. Izračunaj $\frac{\partial z}{\partial x}(1,-1)$ in $\frac{\partial z}{\partial y}(1,-1)$

(ii): $JF = [z\cdot e^{xy}\cdot y \quad z\cdot e^{xy}\cdot x \quad e^{xy}+e^z]$

$JF(1,-1,-1) = [e^{-1} \quad -e^{-1} \quad 2e^{-1}]$

$JF^y = [e^{xy}+e^z] \qquad JF^y(1,-1,-1) = [2e^{-1}]$

$\det JF^y = e^{xy}+e^z$

$\det JF^y(1,-1,-1) = e^{-1}+e^{-1} = 2e^{-1} \neq 0$ ✓

izrek o implicitni funkciji $\rightarrow$ v okolici $(1,-1,-1)$ lahko zapišemo $z=z(x,y)$

1. možnost:

$z\cdot e^{xy}+e^z=0$ / odvajamo po x in upoštevamo $z=z(x,y)$

$z_x\cdot e^{xy} + z\cdot e^{xy}\cdot y + e^z\cdot z_x = 0$

$z_x\cdot(e^{xy}+e^z) = -z\cdot y\cdot e^{xy}$

$$z_x = -\frac{z\cdot y\cdot e^{xy}}{e^{xy}+e^z} \Rightarrow z_x(1,-1) = -\frac{(-1)\cdot(-1)\cdot e^{-1}}{e^{-1}+e^{-1}} = -\frac{\cancel{e^{-1}}}{2\cancel{e^{-1}}} = -\frac{1}{2}$$

$z\cdot e^{xy}+e^z=0$ / odvajamo po y in upoštevamo $z=z(x,y)$

$z_y\cdot e^{xy} + z\cdot e^{xy}\cdot x + e^z\cdot z_y = 0$

$z_y\cdot(e^{xy}+e^z) = -x\cdot z\cdot e^{xy}$

$$z_y = -\frac{z\cdot x\cdot e^{xy}}{e^{xy}+e^z} \Rightarrow z_y(1,-1) = -\frac{(-1)\cdot 1\cdot e^{-1}}{e^{-1}+e^{-1}} = \frac{e^{-1}}{2e^{-1}} = \frac{1}{2}$$

2. možnost

$$Jz(x,y) = \left[\frac{\partial z}{\partial x}(x,y) \quad \frac{\partial z}{\partial y}(x,y)\right]$$

$$Jz(1,-1) = \left[\frac{\partial z}{\partial x}(1,-1) \quad \frac{\partial z}{\partial y}(1,-1)\right]$$

$$= -(JF^{y})^{-1} \cdot JF^{x} =$$

izrek o implicitni preslikavi

$$= -\left[2e^{-1}\right]^{-1} \cdot \left[e^{-1} \quad -e^{-1}\right] =$$

$$= -\frac{e}{2} \cdot \left[e^{-1} \quad -e^{-1}\right] =$$

$$= \left[-\frac{1}{2} \quad \frac{1}{2}\right]$$

$$\Rightarrow \left[\frac{\partial z}{\partial x}(1,-1) \quad \frac{\partial z}{\partial y}(1,-1)\right] = \left[-\frac{1}{2} \quad \frac{1}{2}\right]$$

oz. $\frac{\partial z}{\partial x}(1,-1) = -\frac{1}{2}$ in $\frac{\partial z}{\partial y}(1,-1) = \frac{1}{2}$

Naj bo $f : \mathbb{R}^5 \to \mathbb{R}^2$ podana s predpisom

$$f_1(x_1, x_2, x_3, y_1, y_2) = 2e^{y_1} + y_2 x_1 - 4x_2 + 3$$
$$f_2(x_1, x_2, x_3, y_1, y_2) = y_2 \cos y_1 - 6y_1 + 2x_1 - x_3$$

a) Izračunaj Jacobijevo matriko za f v točki $A(3, 2, 7, 0, 1)$.

b) Ali lahko rešitev $f(x_1, x_2, x_3, y_1, y_2) = 0$ v okolici točke A izrazimo kot $(y_1, y_2) = g(x_1, x_2, x_3)$, torej $f(x_1, x_2, x_3, g(x_1, x_2, x_3)) = 0$?

c) Izračunaj Jacobijevo matriko za g v točki $(3, 2, 7)$.

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a) $$f(x_1,x_2,x_3,y_1,y_2) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$Jf(x_1,x_2,x_3,y_1,y_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & -4 & 0 & 2e^{y_1} & x_1 \\ 2 & 0 & -1 & -y_2 \sin y_1 - 6 & \cos y_1 \end{bmatrix}$$

$$Jf(\overset{x_1}{3},\overset{x_2}{2},\overset{x_3}{7},\overset{y_1}{0},\overset{y_2}{1}) = \begin{bmatrix} 1 & -4 & 0 & 2 & 3 \\ 2 & 0 & -1 & -6 & 1 \end{bmatrix}$$

b) (i) $f(3,2,7,0,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

(ii) $Jf^x = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}$ $\qquad Jf^y = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}$

$$\det Jf^y = 2 \cdot 1 - 3 \cdot (-6) = 2 + 18 = 20 \neq 0$$

$\det Jf^y \neq 0 \xrightarrow{\text{izr. o impl. presl.}}$ rešitev $f(x_1,x_2,x_3,y_1,y_2) = 0$ lahko v okolici $(3,2,7,0,1)$ izrazimo kot $(y_1,y_2) = g(x_1,x_2,x_3)$

c) $$dg(\vec{x}) = Jg(x_1,x_2,x_3) = -(Jf^y)^{-1}(x_1,x_2,x_3,y_1,y_2) \cdot Jf^x(x_1,x_2,x_3,y_1,y_2) =$$

izrek o implicitni preslikavi

$$= -\left(\begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}\right)^{-1} \cdot \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} =$$

$$= -\frac{1}{2 \cdot 1 + 3 \cdot 6} \cdot \begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} = -\frac{1}{20} \cdot \begin{bmatrix} -5 & -4 & 3 \\ 10 & -24 & -2 \end{bmatrix}$$

Naj bo

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

$a, b, c \in \mathbb{R}$ parametri!

a) V okolici katerih točk lahko rešitev izrazimo kot funkcijo $z = z(x, y)$?

b) Dokaži, da v teh točkah velja $x \cdot z \cdot z_x + y \cdot z \cdot z_y = z^2 - c^2$.

c) Naj bo $z = z(x, y)$ rešitev zgornje enačbe v okolici točke $(\frac{a}{2}, \frac{b}{2}, \frac{c}{\sqrt{2}})$; $c \neq 0$,
Izračunaj $z_{xy}(\frac{a}{2}, \frac{b}{2})$.

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a) $F(x,y,z) = \left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right]$

$T(x_0, y_0, z_0)$

Izrek o implicitni preslikavi:

(i): $F(x_0, y_0, z_0) = 0$

(ii): $JF = \left[\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right] \rightarrow JF^Y = \left[\frac{2z}{c^2}\right]$, $\det JF^Y = \frac{2z}{c^2} \neq 0 \rightarrow z \neq 0$

Odgovor: v okolici točk $T(x_0, y_0, z_0)$, $F(x_0, y_0, z_0) = 0$, za katere velja $z_0 \neq 0$

b) $z = g(x,y) = z(x,y)$

$\underline{x \cdot z \cdot z_x + y \cdot z \cdot z_y = z^2 - c^2}$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ / odvajamo po x in upoštevamo $z=z(x,y)$

$\frac{2x}{a^2} + 0 + \frac{2z}{c^2} \cdot z_x = 0 \longrightarrow \frac{x}{a^2} + \frac{z}{c^2} \cdot z_x = 0 \longrightarrow z \cdot z_x = -\frac{x}{a^2} c^2 \quad (*)$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ / odvajamo po y in upoštevamo $z=z(x,y)$

$\frac{2y}{b^2} + 0 + \frac{2z}{c^2} \cdot z_y = 0 \longrightarrow \frac{y}{b^2} + \frac{z}{c^2} \cdot z_y = 0 \longrightarrow z \cdot z_y = -\frac{y}{b^2} c^2 \quad (**)$

$x \cdot z \cdot z_x + y \cdot z \cdot z_y = x \cdot \left(-\frac{x}{a^2} \cdot c^2\right) + y \cdot \left(-\frac{y}{b^2} \cdot c^2\right) = -\frac{x^2}{a^2} c^2 - \frac{y^2}{b^2} \cdot c^2 = -c^2 \cdot \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) =$

$= -c^2 \cdot \left(1 - \frac{z^2}{c^2}\right) = -c^2 + z^2 = z^2 - c^2$

c) $z = z(x,y)$

$T\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{\sqrt{2}}\right)$

$z_{xy}\left(\frac{a}{2}, \frac{b}{2}\right)$

$*$: $z \cdot z_x = -\frac{x}{a^2} \cdot c^2$ / odvajamo po y in upoštevamo $z = z(x,y)$

$z_y \cdot z_x + z \cdot z_{xy} = 0$

$$z_{xy} = -\frac{z_x \cdot z_y}{z} \overset{*}{=} -\frac{1}{z} \cdot \left(-\frac{x}{a^2} \cdot \frac{c^2}{z}\right) \cdot z_y =$$

$$\overset{**}{=} \frac{x}{z^2} \frac{c^2}{a^2} \cdot \left(-\frac{y}{b^2} \cdot \frac{c^2}{z}\right) =$$

$$= -\frac{c^4}{a^2 b^2} \cdot \frac{xy}{z^3}$$

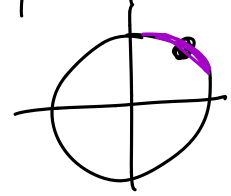
$$z_{xy}\left(\frac{a}{2}, \frac{b}{2}\right) = -\frac{c^{\cancel{4}}}{a^2 b^2} \cdot \frac{\cancel{a} \cdot \cancel{b}}{2 \cdot \cancel{2}} \cdot \frac{\sqrt{2}^{\cancel{3}}}{\cancel{c^3}} = -\frac{\sqrt{2}}{2} \cdot \frac{c}{ab}$$

$x = \frac{a}{2}$

$y = \frac{b}{2}$

$z = z(x,y) = \frac{c}{\sqrt{2}}$

$x^2+y^2=4$



Dan je sistem enačb:

$$y^3 + xz^2 = 8$$
$$xy^2 + z^2 = 9$$

a) Rešite sistem enačb na y in z pri $x = 0$ in $z > 0$.

b) Pokažite, da obstajata taka okolica U točke 0 in taka okolica V rešitve iz prejšnje točke, da je sistem za vse $x \in U$ enolično rešljiv na $(y, z) \in V$. Tako postaneta y in z funkciji spremenljivke x: $y = f(x)$, $z = g(x)$.

c) Izračunajte $f'(0)$ in $g'(0)$.

d) Približno rešite sistem za y in z pri $x = 0.1$, $z > 0$.

---

a) $$y^3 + xz^2 = 8$$
$$xy^2 + z^2 = 9$$

$x=0:\quad y^3 = 8 \longrightarrow y = 2$

$\qquad\quad z^2 = 9 \xrightarrow{z>0} z = 3$

$$\begin{array}{cccc} & \begin{matrix} x \\ N \\ \frac{\partial}{\partial x}\end{matrix} & \begin{matrix} y \\ 0 \\ \frac{\partial}{\partial y}\end{matrix} & \begin{matrix} z \\ 0 \\ \frac{\partial}{\partial z}\end{matrix} \\ JF = & \left[\begin{matrix} z^2 \\ y^2 \end{matrix}\right. & \begin{matrix} 3y^2 \\ 2xy \end{matrix} & \left.\begin{matrix} 2xz \\ 2z \end{matrix}\right] \end{array}$$

b) $$F(x,y,z) = \begin{bmatrix} y^3 + xz^2 - 8 \\ xy^2 + z^2 - 9 \end{bmatrix} \qquad A(0,2,3)$$

(i) $F(0,2,3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark = 0$

(ii) $$JF = \begin{bmatrix} z^2 & 3y^2 & 2xz \\ y^2 & 2xy & 2z \end{bmatrix} \qquad \begin{array}{cccc} & N & 0 & 0 \\ JF(0,2,3) = & \left[\begin{matrix} 9 \\ 4 \end{matrix}\right. & \begin{matrix} 12 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 6 \end{matrix}\right] \end{array}$$

$$JF^x = \begin{bmatrix} 9 \\ 4 \end{bmatrix} \qquad JF^y = \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\det JF^y = 12 \cdot 6 - 0 \cdot 0 = 72 \neq 0$$

$\Rightarrow$ b) sledi iz izreka o implicitni funkciji

c) $\left.\begin{array}{l} y=f(x) \\ z=g(x) \end{array}\right\}$ $G(x)=\begin{bmatrix} f(x) \\ g(x) \end{bmatrix}$ $\quad JG(0)=$ $\quad \begin{bmatrix} y \\ z \end{bmatrix}=G(x)=$

$=\begin{bmatrix} f(x) \\ g(x) \end{bmatrix}$

$$\begin{bmatrix} f'(0) \\ g'(0) \end{bmatrix}=JG(0)=-(JF^{y})^{-1}(0,2,3)\cdot JF^{x}(0,2,3)=$$

$$=-\begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix}^{-1}\cdot\begin{bmatrix} 9 \\ 4 \end{bmatrix}=$$

$$=-\frac{1}{72}\begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix}\cdot\begin{bmatrix} 9 \\ 4 \end{bmatrix}=$$

$$=-\frac{1}{12}\cdot\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\cdot\begin{bmatrix} 9 \\ 4 \end{bmatrix}=$$

$$=-\frac{1}{12}\cdot\begin{bmatrix} 9 \\ 8 \end{bmatrix}=\begin{bmatrix} -\frac{3}{4} \\ -\frac{2}{3} \end{bmatrix}$$

$\Rightarrow f'(0)=-\frac{3}{4},\ g'(0)=-\frac{2}{3}$

* druga možnost: odvajanje sistema po x in upostevanju $y=y(x)$, $z=z(x)$ za $x=0$, $y=2$ in $z=3$.

d)

$G(0.1)\approx G(0)+JG(0)\cdot[0.1-0]=$ $\quad \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}=G(x)\approx G(0)+JG(0)\cdot[x-0]$

$\begin{bmatrix} y(0.1) \\ z(0.1) \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \end{bmatrix}+\begin{bmatrix} -\frac{3}{4} \\ -\frac{2}{3} \end{bmatrix}\cdot[0.1]=$

$$=\begin{bmatrix} 2-\frac{3}{4}\cdot\frac{1}{10} \\ 3-\frac{2}{3}\cdot\frac{1}{10} \end{bmatrix}=\begin{bmatrix} \frac{80}{40}-\frac{3}{40} \\ \frac{45}{15}-\frac{1}{15} \end{bmatrix}=\begin{bmatrix} \frac{77}{40} \\ \frac{44}{15} \end{bmatrix}=\begin{bmatrix} 1.925 \\ 2.9\overline{3} \end{bmatrix}$$

$f(0.1)\doteq 1.925 \quad g(0.1)\doteq 2.9\overline{3}$

Naj bo $f : \mathbb{R}^6 \to \mathbb{R}^2$ podana s predpisom

$$f_1(x_1, x_2, x_3, x_4, y_1, y_2) = x_1x_2x_4 + y_1x_3 - 4y_2 + 10$$
$$f_2(x_1, x_2, x_3, x_4, y_1, y_2) = x_1 + x_2 + y_1^2 - 2y_2 - 5x_3x_4 + y_1y_2x_3 - 9$$

a) Pokaži, da lahko v okolici točke $A(1, 0, 1, 1, 2, 3)$ rešitev enačbe $f = 0$ zapišemo kot $(y_1, y_2) = g(x_1, x_2, x_3, x_4)$.

b) Izračunaj $Jg(1, 0, 1, 1)$.

---

a) $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

(i): $f(1,0,1,1,2,3) = \begin{bmatrix} 0+2-12+10 \\ 1+0+2^2+3-5+6-9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(ii): $Jf = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} x_2x_4 & x_1x_4 & y_1 & x_1x_2 & x_3 & -4 \\ 1 & 1 & -5x_4+y_1y_2 & -5x_3 & 2y_1+y_2x_3 & 1+y_1x_3 \end{bmatrix}$

$Jf(1,0,1,1,2,3) = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & -4 \\ 1 & 1 & 1 & -5 & 7 & 3 \end{bmatrix}$

$Jf^y = \begin{bmatrix} 1 & -4 \\ 7 & 3 \end{bmatrix}$

$\det Jf^y = 1 \cdot 3 - (-4) \cdot 7 = 3 + 28 = 31 \neq 0$

b) $(y_1, y_2) = g(x_1, x_2, x_3, x_4)$

$Jg(1,0,1,1) = -(Jf^y)^{-1} \cdot Jf^x = - \begin{bmatrix} 1 & -4 \\ 10 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & -5 \end{bmatrix} =$

$= -\frac{1}{31} \cdot \begin{bmatrix} 3 & 4 \\ -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & -5 \end{bmatrix} =$

$= -\frac{1}{31} \cdot \begin{bmatrix} 4 & 7 & 10 & -20 \\ 1 & -6 & -13 & -5 \end{bmatrix}$

Dana je enačba $e^{xz} - y - 5z = 0$.

a) Rešite enačbo na z pri $x = y = 0$.

b) Pokažite, da obstajata taka okolica U izhodišča in taka okolica V rešitve iz prejšnje točke, da je enačba za vse $(x, y) \in U$ enolično rešljiva na $z \in V$. Tako postane z funkcija spremenljivk x in y: $z = f(x, y)$.

c) Izračunajte $f_x(0, 0)$ in $f_y(0, 0)$.

d) Približno rešite enačbo na z pri $x = 0.1$, $y = 0.2$.

---

a) $x=y=0$: $\quad e^{0\cdot z} - 0 - 5\cdot z = 0 \longrightarrow 1-5z=0 \longrightarrow z = \frac{1}{5} = 0{,}2$

b) $F(x,y,z) = [e^{xz} - y - 5z]$

(i): $F(0,0,0{,}2) = [0]$ ✓

(ii): $JF = [e^{x\cdot z}\cdot z \quad -1 \quad e^{xz}\cdot x - 5]$

$JF(0,0,0{,}2) = [0{,}2 \quad -1 \quad -5]$

$JF^{y} = [-5]$

$\det JF^{y} = -5 \neq 0$

$\rightarrow$ b) velja po izreku o implicitni preslikavi

$z = f(x,y)$ v okolici $(0,0)$

c) $[f_x(0,0) \quad f_y(0,0)] = Jf(0,0)$

$$Jf(0,0) \underset{\substack{\uparrow \\ \text{izrek} \\ \text{o imp. pre.}}}{=} -(JF^{y})^{-1}\cdot JF^{x} = -\frac{1}{-5}\cdot[0{,}2, -1] = \frac{1}{5}\cdot[0{,}2 ; -1] = [\tfrac{1}{25} ; -\tfrac{1}{5}] = [0{,}04, -0{,}2]$$

d) $$z = f(0{,}1, 0{,}2) \approx f(0,0) + JF(0,0)\cdot\begin{bmatrix}0{,}1-0\\0{,}2-0\end{bmatrix} =$$

$$= 0{,}2 + [0{,}04, -0{,}2]\cdot\begin{bmatrix}0{,}1\\0{,}2\end{bmatrix} =$$

$$= 0{,}2 + 0{,}004 - 0{,}04 = \underline{\underline{0{,}164}}$$