$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$N \cdot N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N = \begin{array}{c} \left[\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}\right] \end{array} \begin{array}{l} e_2 \\ e_3 \\ e_4 \\ 0 \end{array} \qquad \begin{array}{cccc} 0 & e_1 & e_2 & e_3 \\ \hline \end{array}$$

$$\begin{array}{cccc} 0 & e_1 & e_2 & e_3 \end{array}$$
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} e_3 \\ e_4 \\ 0 \\ 0 \end{array} \qquad N^3 = \begin{bmatrix} & & 1 \\ & 0 & \\ & & \end{bmatrix}, \quad N^4 = 0$$

N: nilpotent $\qquad$ N: $n \times n$ matrika, $N^n = 0$.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{N} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$I \cdot B = B = BI$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

$$A = I + N$$

$$A^n = (I + N)^n = I^n + \overset{n}{\overset{\parallel}{\binom{n}{1}}} I^{n-1} \cdot N + \binom{n}{2} I^{n-2} \cdot \overset{0}{\overset{\parallel}{N^2}} + \dots$$

$$= I + n \cdot N = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = 2 \cdot I + N$$

$$A^n = (2I + N)^n = \overset{2^n I}{\overset{\parallel}{(2I)^n}} + \binom{n}{1}(2I)^{n-1} \cdot N + \binom{n}{2}(2I)^{n-2} N^2 +$$

$$+ \underbrace{\binom{n}{3}(2I)^{n-3} N^3}_{\overset{\parallel}{0}} + \underbrace{\dots}_{=0}$$

$$= \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{bmatrix} + n \cdot 2^{n-1} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{n(n-1)}{2} 2^{n-2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = c \cdot B \qquad c \in \mathbb{R}$$

$$A^2 = c \cdot B \cdot c \cdot B = c^2 \cdot B^2$$

Pomembno: $A = D = \operatorname{diag}\{d_1, \dots, d_n\}$; $D^k = \operatorname{diag}\{d_1^k, d_2^k, \dots, d_n^k\}$

$$D^2 = \begin{array}{c} e_1 \\ e_2 \\ \vdots \\ e_n \end{array} \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix} \cdot \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_n \end{array} \\ \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix} \end{array} =$$

$$= \begin{bmatrix} d_1^2 & & & \\ & d_2^2 & & 0 \\ & & \ddots & \\ & 0 & & d_n^2 \end{bmatrix}$$

ničle nad diagonalo

ničle pod diagonalo.

$$\begin{array}{c} e_1 \\ \vdots \\ e_n \end{array} \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{array}{c} \begin{array}{ccc} e_1 & \cdots & e_n \end{array} \\ \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix} \end{array} = \begin{bmatrix} d_1 c_1 & & & 0 \\ & d_2 c_2 & & \\ & & \ddots & \\ 0 & & & d_n c_n \end{bmatrix}$$

Transponiranje: $\underset{m \times n}{A} \rightarrow \underset{n \times m}{A^T}$ vrstice napišemo v stolpce

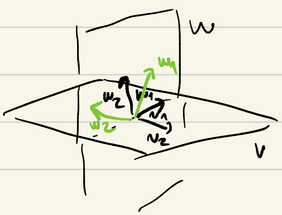
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

---

np: V, W; $V + W$: v.p.

$V + W = \mathbb{R}^3$ $\dim V = \dim W = 2$

$\mathbb{R}^3$



$\dim V + \dim W = 4$

$V \cap W$: šteli $2\times$

baza V: $\{v_1, v_2\}$

baza W: $\{w_1, w_2\}$ $\{w_1, w_2\}$ $w_1 \neq v_1$

$V + W = \mathrm{Lin}\{v_1, v_2, \cancel{w_1}, w_2\}$

$V + W = \mathrm{Lin}\{\underbrace{v_1, v_2, w_1}_{\text{lin. neodv.}}, w_2\}$

lin. neodv → ga dodam, sicer izpustim in gledam naslednjega

linearne preslikave

<u>Def</u>: Naj bosta V, W vektorska prostora. Preslikava

$\mathcal{A}: V \to W$ je linearna, če

$$\mathcal{A}(\alpha v_1 + \beta v_2) = \alpha \mathcal{A}(v_1) + \beta \mathcal{A}(v_2).$$

Primeri.

Ugotovi, ali so preslikave linearne:

$$\mathcal{A}: \mathbb{R}^2 \to \mathbb{R}^3 \qquad \mathcal{A}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [x+y,\ 2x+3y,\ x-y]^T$$

$$\underbrace{\mathcal{A}\left(\alpha \begin{bmatrix} x \\ y \end{bmatrix} + \beta \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)} = \mathcal{A}\left(\begin{bmatrix} \alpha x + \beta x_1 \\ \alpha y + \beta y_1 \end{bmatrix}\right) =$$

$$= \left((\alpha x + \beta x_1) + (\alpha y + \beta y_1),\ 2(\alpha x + \beta x_1) + 3(\alpha y + \beta y_1),\ (\alpha x + \beta x_1) - (\alpha y + \beta y_1)\right)^T$$

$$= (\alpha(x+y),\ 2\alpha x + 3\alpha y,\ \alpha(x-y))^T + (\beta x_1 + \beta y_1,\ 2\beta x_1 + 3\beta y_1,\ \beta x_1 - \beta y_1)^T =$$

$$= \alpha(x+y,\ 2x+3y,\ x-y)^T + \beta \cdot (x_1+y_1,\ 2x_1+3y_1,\ x_1-y_1)^T$$

$$= \underbrace{\alpha \mathcal{A}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + \beta \mathcal{A}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)}$$

$$\underbrace{e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\mathcal{A}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [x+y,\ 2x+3y,\ x-y]^T$$

$$\mathcal{A}(x \cdot e_1 + y \cdot e_2) = x \cdot \mathcal{A}(e_1) + y \cdot \mathcal{A}(e_2) =$$

$$= x \cdot [1, 2, 1]^T + y \cdot [1, 3, -1]^T = \underbrace{(\text{v st. bazi})}$$

$$= A \to \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x+3y \\ x-y \end{bmatrix}$$

$\mathcal{A}$: preslikava

A: matrika, ki pripada $\mathcal{A}$, ko smo fiksirali baze

Naj bo v $\mathbb{R}^2$ baza $\{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

v $\mathbb{R}^3$: std. baza $\{e_1, e_2, e_3\}$

Naj bo $v \in \mathbb{R}^2$ polj. vektor. Potem $\exists\ \alpha, \beta$:

$v = \alpha v_1 + \beta v_2$

$$\mathcal{A}(v) = \mathcal{A}(\alpha v_1 + \beta v_2) = \alpha \underline{\mathcal{A}(v_1)} + \beta \underline{\mathcal{A}(v_2)} =$$

$$= \alpha \cdot \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 0 \\ +1 \\ -2 \end{bmatrix} = B = \begin{bmatrix} 2 & 0 \\ 5 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

matrika B pripada $\mathcal{A}$ v bazah $\{v_1, v_2\}$ za $V = \mathbb{R}^2$, $\{e_1, e_2, e_3\}$ za $W = \mathbb{R}^3$.

Izrek: Naj bo $\mathcal{A}: V \to W$ linearna preslikava. Potem je $\mathcal{A}$ natanko določena s slikami baznih vektorjev.

TOREJ: da podamo $\mathcal{A}$, ki slika iz V - n-dim v. p., je dovolj, da povemo slike baznih vektorjev V, V podamo na n lin. neodv. baznih vektorjih.

Dani linearni preslikavi $\mathcal{A}$ pripadajo pri različnih bazah V, W različne matrike!

CILJ: Najti čimboljše baze za linearne preslikave.

Čimboljše: A je diagonalna ali

A ima nad diagonalo kvečjemu 1 vrsto.

Ali je preslikava $\mathcal{A}$ linearna?

$$\mathcal{A}: \mathbb{R}^3 \to \mathbb{R}^2 \quad \mathcal{A}([x, y, z]^T) = [x + y - z, 2 + y]^T$$

$$\| $$

$$x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + z \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Koliko je $\mathcal{A}(0)$, če je $\mathcal{A}$ linearna?

Dobimo iz $\mathbb{R}^2$, vendar $\mathcal{A}$ ni linearna, ker ne spoštuje lin. kombinacij

3. Pokaži, da je preslikava

$$\mathcal{A}: \mathbb{R}^4 \to \mathbb{R}, \quad \mathcal{A}([x,y,z,w]^T) = x - y + 2z - 6w$$

linearna in jo zapiši z matriko v standardnih bazah.

$$\mathcal{A}(xe_1 + ye_2 + ze_3 + we_4) = x \cdot \underset{\substack{\parallel \\ 1}}{\mathcal{A}(e_1)} + y \cdot \underset{\substack{\parallel \\ -1}}{\mathcal{A}(e_2)} + z \cdot \underset{\substack{\parallel \\ 2}}{\mathcal{A}(e_3)} + w \underset{\substack{\parallel \\ -6}}{\mathcal{A}(e_4)}$$

$$A = \begin{bmatrix} 1 & -1 & 2 & -6 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = x - y + 2z - 6w$$

$$a = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -6 \end{bmatrix}$$

Lin. preslikava $\mathcal{A}: \mathbb{R}^n \to \mathbb{R}$ se imenuje <u>linearni funkcional</u>. V std bazah mu pripada matrika A. Potem je

$\mathcal{A}(x) = A \cdot x = \langle a, x \rangle$, kjer je

$a = A^T$: vektor. Ta vektor se

imenuje GRADIENT $\mathcal{A}$.

<u>Gradient</u> pove smer max naraščanja.

$$\mathcal{A}(x) = \langle a, x \rangle = |a| \cdot |x| \cdot \cos\varphi \qquad |x| = 1$$

Pri katerem x, $|x| = 1$, je $\mathcal{A}(x)$ maksimalna? $\varphi = 0 \Rightarrow$

$x = \lambda \cdot a$, $\lambda > 0 \Rightarrow a$ je smer max naraščanja.

Če je $f: \mathbb{R}^n \to \mathbb{R}$ pojutna zv. odv. preslikava (std. baze!).

Potem je $\operatorname{grad} f(x_0)$ enak gradientu LINEARNE preslikave, ki

v točki x_0 aproksimira $f(x) - f(x_0) = A \cdot (x - x_0) + o(|x - x_0|)$

$\uparrow$ to je iskani $\mathcal{A}$

$$A = [f_{x_1}, f_{x_2}, \dots f_{x_n}] \;; \quad \operatorname{grad} f(x_0) = [f_{x_1}, \dots, f_{x_n}]^T = (f_{x_1}, \dots, f_{x_n})$$

Naj bo $\mathcal{A}: V^n \to W^m$ dana linearna preslikava.

- Potem je $\operatorname{Im} \mathcal{A}$ vektorski podprostor W. Njegova dimenzija se imenuje RANG preslikave $\mathcal{A}$.
- Prostor $\operatorname{Ker} \mathcal{A} := \mathcal{A}^{-1}(0)$ je (tudi) vektorski podprostor prostora V.
  ↳ kernel

→ $\operatorname{Im} \mathcal{A} = \operatorname{Lin}\{\mathcal{A}(v_1), \ldots, \mathcal{A}(v_n)\}$, kjer so $v_1, \ldots, v_n$ baza V.
$\dim \operatorname{Im} \mathcal{A} \leq \min\{m, n\}$

→ $\mathcal{A}^{-1}(0)$ je vektorski podprostor V.

Vemo: $0 \in \operatorname{Ker} \mathcal{A}$.

Denimo, da $v, w \in \operatorname{Ker} \mathcal{A}$.

$$\mathcal{A}(\alpha v + \beta w) = \alpha \underset{\overset{\shortparallel}{0}}{\mathcal{A}(v)} + \beta \underset{\overset{\shortparallel}{0}}{\mathcal{A}(w)} = 0 \implies \alpha v + \beta w \in \operatorname{Ker} \mathcal{A}.$$