$$f(x) = \frac{x+1-\sqrt{x+1}}{x}$$

Določi: asimpt -

Ali obstaja

Asimptote f? "robovi def. območja od f"

$$D_f = \cancel{\mathbb{R} \setminus \{0\}} = [-1,0) \cup (0,\infty)$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x+1-\sqrt{x+1}}{x} = \lim_{x\to 0} 1 + \frac{1}{x} - \frac{\sqrt{x+1}}{x} =$$

$$= 1 + \lim_{x\to 0} \underbrace{\frac{1}{x}}_{\downarrow\ \infty} - \underbrace{\frac{\sqrt{x+1}}{x}}_{\downarrow\ \pm\infty} = 1 - \frac{1}{2} = \underline{\underline{\frac{1}{2}}}$$

$$\lim_{x\downarrow 0} \frac{1}{x} - \frac{\sqrt{x+1}}{x} = \lim_{x\downarrow 0} \frac{1-\sqrt{x+1}}{x} \cdot \frac{(1+\sqrt{x+1})}{(1+\sqrt{x+1})} = \lim_{x\downarrow 0} \frac{1-(x+1)}{x(1+\sqrt{x+1})} =$$

$$= \lim_{x\downarrow 0} - \frac{\cancel{x}}{\cancel{x}(1+\sqrt{x+1})} = -\frac{1}{2} = \lim_{x\uparrow 0} \frac{1}{x} - \frac{\sqrt{x+1}}{x}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+1-\sqrt{x+1}}{x} = 1$$

$$\lim_{y\to 0} y \overbrace{\left(\frac{y+1}{y} - \sqrt{\frac{y^2+1}{y^2}}\right)}^{g(y)} = ?$$

$y=0$

$$\lim_{y\downarrow 0} y \left(\frac{y+1}{y} - \sqrt{\frac{y^2+1}{y^2}}\right) = \lim_{y\downarrow 0} \cancel{y} \cdot \frac{y+1-\sqrt{y^2+1}}{\cancel{y}} = \underline{\underline{0}}$$

$|y>0$

sk. imenovalec

$\sqrt{y^2} = y$

$y \in (-1, 0)$

$$\lim_{y \nearrow 0} y \left(\frac{y+1}{y} - \sqrt{\frac{y^2+1}{y^2}} \right) = \lim_{y \nearrow 0} \cancel{y} \frac{y+1+\sqrt{y^2+1}}{\cancel{y}} = \underline{\underline{2}}$$

$y < 0$

$\sqrt{y^2} = -y$

Sklep: $\lim_{y \to 0} g(y)$ ne obstaja.

$$\lim_{y \to 0} y \left(\underbrace{\frac{y+1}{y} - \sqrt{\frac{y^2+1}{y^2}}}_{\text{sk. imen.}} \right) = \lim_{y \to 0} y \frac{y+1}{y} - \lim_{y \to 0} y \sqrt{\frac{y^2+1}{y^2}}$$

$\sqrt{y^2}$

ZVEZNOST / ENAKOMERNA ZVEZNOST

f je zvezna v $x \in D_f$, če:

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x' \in D_f : |x - x'| < \delta \quad |f(x) - f(x')| < \varepsilon$

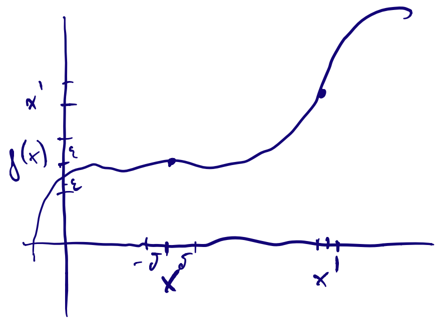
f je zvezna, če je je zvezna v vsaki točki iz D_f.

$\forall x \in D_f\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x' \in D_f : |x - x'| < \delta,\ |f(x) - f(x')| < \varepsilon$

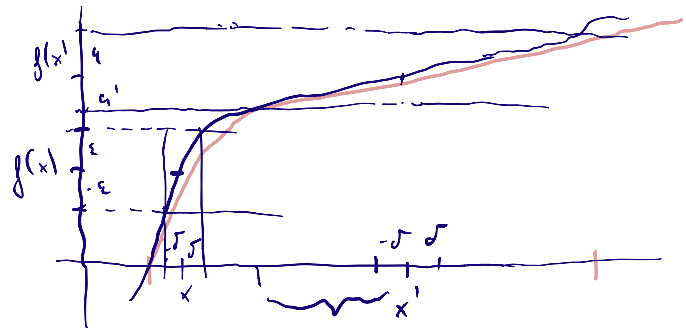
f je enakomerno zvezna na D_f, če velja:

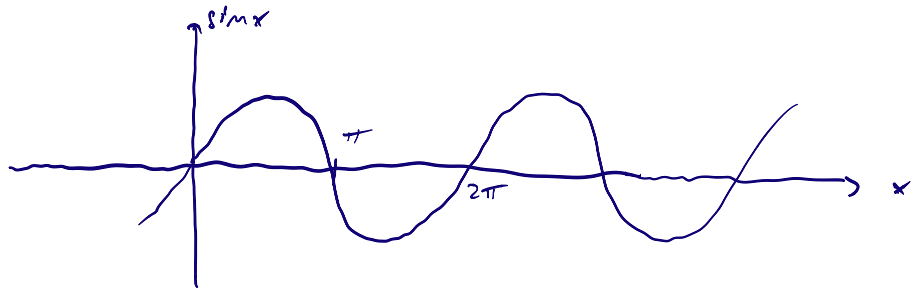
$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D_f\ \forall x' \in D_f : |x - x'| < \delta;\ |f(x) - f(x')| < \varepsilon$

ZVEZNOST



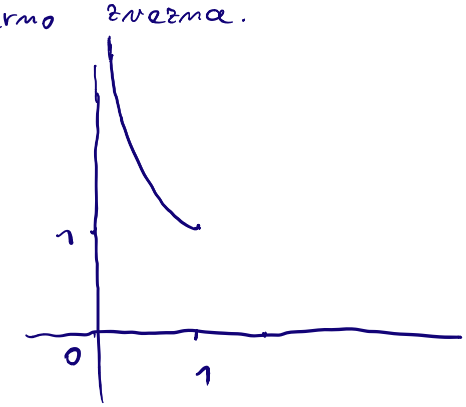
ENAKOMERNA ZVEZNOST





Trditev: Če je f zvezna na zaprtem intervalu, potem je enakomerno zvezna.

$f(x) = \frac{1}{x}$ na $(0,1)$

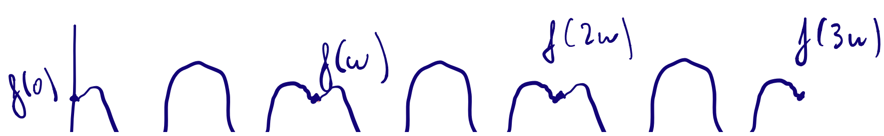


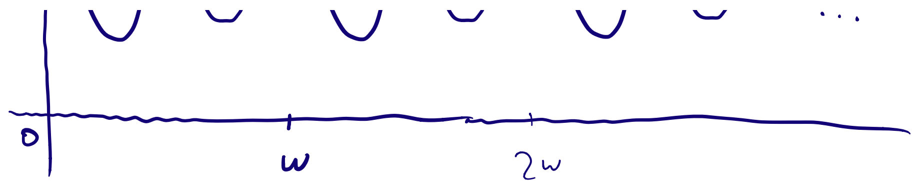
$f: \mathbb{R} \longrightarrow \mathbb{R}$

Trditev: f je zvezna in periodična. Dokaži, da je enakomerno zvezna.

f periodična, potem $\exists \omega > 0$, da velja:

$$f(x) = f(x+\omega) \quad \text{za vsak } x \in \mathbb{R}$$





Na $[0,w]$ je f enakomerno zvezna.

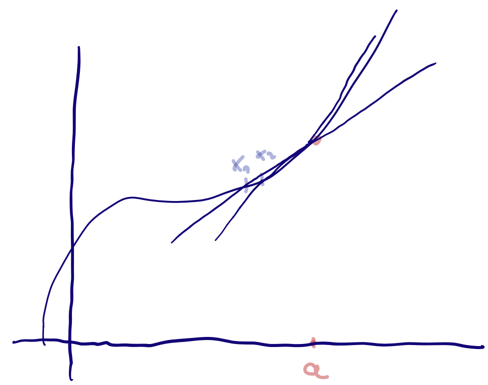
Torej $\forall \varepsilon>0\ \exists \delta>0\ \forall x\in[0,w]\ \forall x'\in[0,w]:$

$$|x-x'|<\delta \Rightarrow |f(x)-f(x')|<\varepsilon$$

$\forall \varepsilon>0\ \exists \delta>0\ \forall x\in\mathbb{R}\ \forall x'\in\mathbb{R}:$

$$|x-x'|<\delta \Rightarrow |f(x)-f(x')|<\varepsilon$$

ODVODI



$a \in D_f$

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$f(x) \approx \overset{f'(a)}{\overset{\shortparallel}{k}}(x-a)+\underset{f(a)}{\underset{\shortparallel}{n}}$$

"kako f predstaviti kot funkcijo, ki je v okolici točke a podobna linearni funkciji"

Izračunaj odvod funkcije.

$$f(x)=\frac{2^{3x}}{3^{x^2}}$$

$$\left(\frac{2^{3x}}{3^{x^2}}\right)' = \frac{(2^{3x})' 3^{x^2} - 2^{3x}\cdot(3^{x^2})'}{(3^{x^2})^2} =$$

$$= \frac{2^{3x}\cdot \ln 2 \cdot 3 \cdot 3^{x^2} - 2^{3x}\, 3^{x^2}\cdot \ln 3 \cdot 2x}{3^{x^2}\cdot 3^{x^2}} =$$

$$= \frac{2^{3x}\,\cancel{3^{x^2}}\,(\ln 8 - x \ln 9)}{3^{\cancel{2}x^2}}$$

$$f(x) = \sin x^{\tan x} = e^{\ln(\sin x^{\tan x})} = e^{\tan x \ln(\sin x)}$$

$$\left[e^{\tan x \ln(\sin x)}\right]' = e^{\tan x \ln(\sin x)}\left(\tan x \cdot \ln(\sin x)\right)' =$$

$$= e^{\tan x \ln(\sin x)}\left[\frac{\ln(\sin x)}{\cos^2 x} + \underbrace{\tan x \cdot \frac{\cos x}{\sin x}}_{1}\right] =$$

$$= \sin^{\tan x}\left(\frac{\ln(\sin x)}{\cos^2 x} + 1\right)$$

$$f(x) = \ln(\overbrace{x + \sqrt{x^2+1}}^{g(x)}) = \underbrace{\frac{1}{x + \sqrt{x^2+1}}}_{\ln'(g(x))} \cdot \underbrace{\left(1 + \frac{1}{\cancel{2}\sqrt{x^2+1}}\,\cancel{2}x\right)}_{g'(x)} =$$

$$(\ln x)' = \frac{1}{x}$$

$$= \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{x}{\sqrt{x^2+1}}\right) =$$

$$= \frac{1}{\cancel{x+\sqrt{x^2+1}}} \cdot \frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}} = \underline{\underline{\frac{1}{\sqrt{x^2+1}}}}$$

$$\left(x+\sqrt{x^2+1}\right)' = 1 + \frac{1}{2}\frac{\overbrace{2x}^{(x^2+1)'}}{\sqrt{x^2+1}}$$

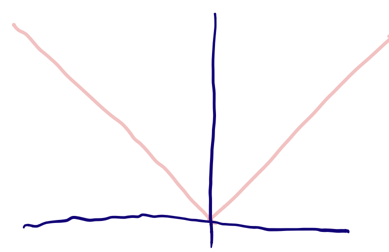
$f(x) = |x|$

f' ne obstaja

$$\lim_{x \uparrow 0} \underbrace{\frac{f(x)}{x}} = -1 \quad ; \quad \lim_{x \downarrow 0} \frac{f(x)}{x} = 1$$

↳ $x < 0$; $x > 0$

↳ $\dfrac{f(x) - \overset{=0}{f(0)}}{\underset{=0}{x - 0}}$

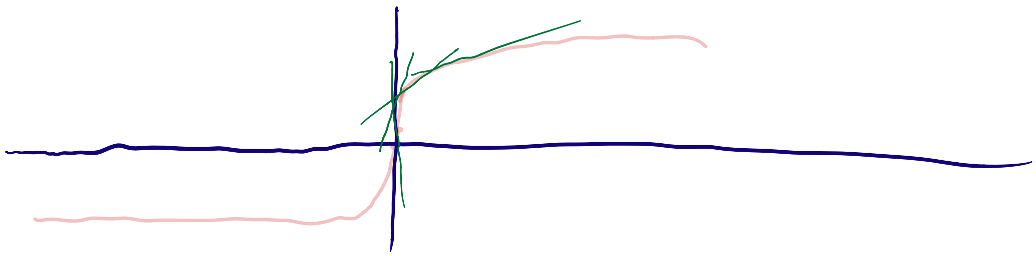


$f(x) = x|x|$

$$\lim_{x \to 0} \frac{x|x| - 0}{x - 0} = \lim_{x \to 0} \frac{\cancel{x}|x|}{\cancel{x}} = \lim_{x \to 0} |x| = 0$$

↳ funkcija je odvedljiva v 0

$f(x) = \sqrt[3]{x}$, f ni odvedljiva v 0



$$f'(x) = \underset{x \neq 0}{\underbrace{\tfrac{1}{3} x^{-\frac{2}{3}}}}$$, ne moremo vstaviti $x=0$, ker bi delili z nič

$$f'(0) = \lim_{x \to 0} \frac{\sqrt[3]{x}}{x}$$ ... ne obstaja

$$g(x) = \frac{1}{2} \arctan\left(\frac{2x}{1-x^2}\right)$$

$$g'(x) = \frac{1}{\cancel{2}} \frac{1}{1 + \left(\frac{2x}{1-x^2}\right)^2} \cdot \frac{\cancel{2} \cdot (1-x^2) + \cancel{2}x \cdot 2x}{(1-x^2)^2} =$$

$$= \frac{\cancel{(1-x^2)^2}}{(1-x^2)^2 + (2x)^2} \cdot \frac{1-x^2 + \overbrace{2x^2}^{2x^2}}{\cancel{(1-x^2)^2}} =$$

$$= \frac{1}{\underbrace{1 - 2x^2 + x^4 + 4x^2}_{x^4 + 2x^2 + 1}} \cdot \frac{1+x^2}{1} =$$

$$= \frac{1}{(x^2+1)^2} \cdot \frac{1+x^2}{1} = \frac{1}{1+x^2} = \operatorname{arctg}'(x)$$

Skiciraj graf g.

$f, g : \mathbb{R} \longrightarrow \mathbb{R}$, $\forall x \in \mathbb{R} : f'(x) = g'(x)$

$\Rightarrow \exists C \in \mathbb{R} : g(x) = f(x) + C$

$D_g = \mathbb{R} \setminus \{-1, 1\}$

- $x < -1$ ... $g(x) = \arctan(x) + C_1$ *
- $x \in (-1, 1)$ ... $g(x) = \arctan(x) + C_2$
- $x > 1$ ... $g(x) = \arctan(x) + C_3$

Kako dobimo C_1, C_2, C_3 ?

- $x < -1$

$x' < -1$

$g(x') = \arctan(x') + C_1$ (1 način)

Drug način:

$* \Rightarrow \lim\limits_{x \to -\infty} g(x) = \lim\limits_{x \to -\infty} \arctan(x) + C_1$

$\lim\limits_{x \to -\infty} \frac{1}{2} \cdot \text{arctg}\left(\frac{2x}{1-x^2}\right) = 0$

$\Big\downarrow x \to -\infty$

$\frac{2x}{1-x^2} = \frac{2}{\frac{1}{x} - x}$, $\frac{1}{x} \to 0$, $x \to \infty$, $\to 0$

$\lim\limits_{x \to -\infty} \arctan x = -\frac{\pi}{2}$

$\Rightarrow$

$0 = -\frac{\pi}{2} + C_1$

$C = \frac{\pi}{2}$

- $x \in (-1, 1)$

$$g(x) = \arctan x + C_2$$

$$= \frac{1}{2}\arctan\left(\frac{2x}{1-x^2}\right)$$

$x = 0$:

$$0 = 0 + C_2 \Rightarrow \underline{\underline{C_2 = 0}}$$

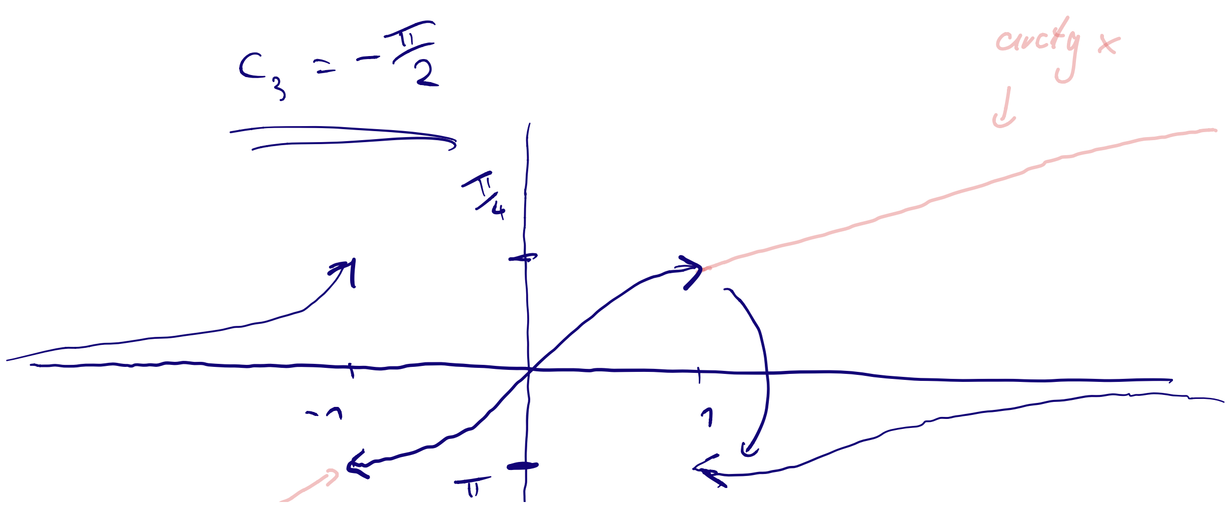
- $x > 1$ $\quad C_3$?

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{1}{2}\arctan\left(\underbrace{\frac{2x}{1-x^2}}_{\substack{\downarrow\, x\to\infty \\ 0}}\right) = 0$$

$$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$$

$$0 = \frac{\pi}{2} + C_3$$

$$\underline{\underline{C_3 = -\frac{\pi}{2}}}$$



odvedljiva

f soda, $f: \mathbb{R} \to \mathbb{R}$. Dokaži, da je f' liha funkcija

$\forall a \in \mathbb{R}: \quad f'(-a) = -f'(a)$ (LIHOST) $\to$ To želimo dokazat

$f(a) = f(-a) \quad \forall a \in \mathbb{R}$

↳ To vemo

$f(x) = f(-x)$, saj je f soda

$f(-a) = f(a)$

po definiciji odvoda:

$$f'(-a) = \lim_{x \to -a} \frac{f(x) - f(-a)}{x - (-a)} = \lim_{x \to -a} \frac{f(-x) - f(a)}{x + a} =$$

$$= \lim_{x \to -a} \frac{f(-x) - f(a)}{x + a} = \lim_{x \to -a} \frac{f(-x) - f(a)}{-(-x - a)} =$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

??

$$= -\lim_{x \to -a} \frac{f(-x) - f(a)}{-x - a} = -\underbrace{\lim_{t \to a} \frac{f(t) - f(a)}{t - a}}_{f'(a)}$$

$-x =: t$

Dokazali smo, da velja: $f'(-a) = -f'(a)$

Podobno: odvod lihe funkcije je sod.

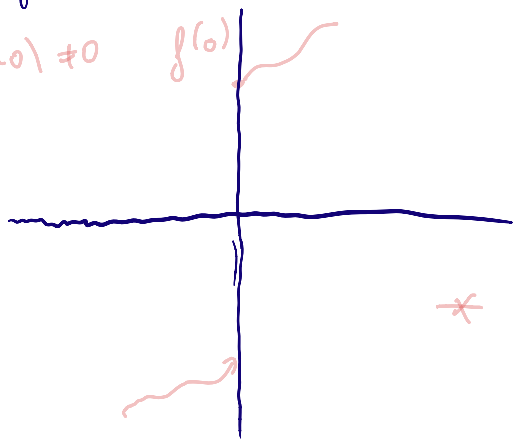
f je soda, $f \in C^{\infty}(\mathbb{R})$

$f^{(2n-1)}(0) = 0$ za vse $n \geq 1$

0

Možnost: f zvezna in liha $\Rightarrow f(0)=0$

če $f(0)\neq 0$



f soda $\Rightarrow f'$ liha $\Rightarrow f''$ soda $\Rightarrow f'''$ liha ...

$f^{(2n-1)}$ je liha funkcija za vse $n\geq 1$, je zvezna (saj je odvedljiva) $\Rightarrow f^{(2n-1)}(0)=0$

f odvedljiva v a $\Rightarrow$ f zvezna v a

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{\underbrace{x-a}_{\to 0}} \in \mathbb{R}$$

$f(x)-f(a) \to 0$

$\hookrightarrow f$ odv. v a

$f,g\in C^{\infty}(\mathbb{R})$. Dokaži:

$$(fg)^{n}=\sum_{k=0}^{n}\binom{n}{k} f^{(n-k)} g^{(k)} \quad \text{za vsak } n$$

$n=1$:

$(fg)'=f'g+g'f$ ✓

$n \Rightarrow n+1$

$$(fg)^{(n+1)} = \left[(fg)^{(n)}\right]' \overset{\text{I.P.}}{=}$$

$$= \left[\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}\right]' =$$

pravilo za odv. produkta

$$= \sum_{k=0}^{n} \binom{n}{k} \underbrace{\left[f^{(n+1-k)} g^{(k)} + f^{(n-k)} g^{(k+1)}\right]}_{\left[f^{(n-k)} g^{(n)}\right]'} = ?$$

$\hookrightarrow k = n$

$$f^{(2)}(x) = f''(x) = \left[f'(x)\right]'$$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} \overbrace{f^{(n+1-k)} g^{(k)}} = ?$$

$k = 0, \ldots, n+1$

$f^{(n+1-k)} g^{(k)}$ ... gledamo koeficiente pred tem produktom v zg. vsoti

$k = 0, \ldots, n$:

$$\binom{n}{k}$$

$$\underbrace{\qquad}_{n+k}$$

$$f^{(n-k)} g^{(k+1)} = f^{(n+1-\underbrace{(k+1)}_{l})} g^{\underbrace{(k+1)}_{l}} \quad (k \mapsto k-1)$$

$$\binom{n}{k+1}$$

$$f^{(n+1-l)} g^{(l)}$$

$$\binom{n}{k+1}$$

$l \mapsto k+1$

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} = \binom{n+1}{l}$$

$k = 0, \ldots, n$:

Koeficient pred $f^{(n+1-k)} g^{(k)}$: $\binom{n+1}{k}$

koef. pred $f^{(n+1-n+1)} g^{(n+1)}$ ... 1

L'HÔPITALOVO PRAVILO

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ ... če sta f in g odvedljivi

↳ $a \in \mathbb{R} \cup \{-\infty, \infty\}$

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$$

ALI

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{IN} \quad \lim_{x \to a} f(x) = \pm\infty$$

$$\lim_{x \downarrow 0} x^x = \lim_{x \downarrow 0} e^{x \ln x}$$

$$\lim_{x \downarrow 0} \underset{\substack{\downarrow \\ 0}}{x} \cdot \underset{\substack{\downarrow \\ -\infty}}{\ln x} = \lim_{x \downarrow 0} \frac{\overset{-\infty}{\ln x}}{\underset{\substack{\downarrow \\ \infty}}{\frac{1}{x}}} \overset{L'H}{=} \lim_{x \downarrow 0} \frac{\frac{1}{\cancel{x}}}{-\frac{1}{x^{\cancel{2}}}} = \lim_{x \downarrow 0} -x$$

$$= \underline{\underline{0}}$$

$$\lim_{x \downarrow 0} e^{x \ln x} = e^{\lim_{x \downarrow 0} x \ln x} = e^0 = \underline{\underline{1}}$$