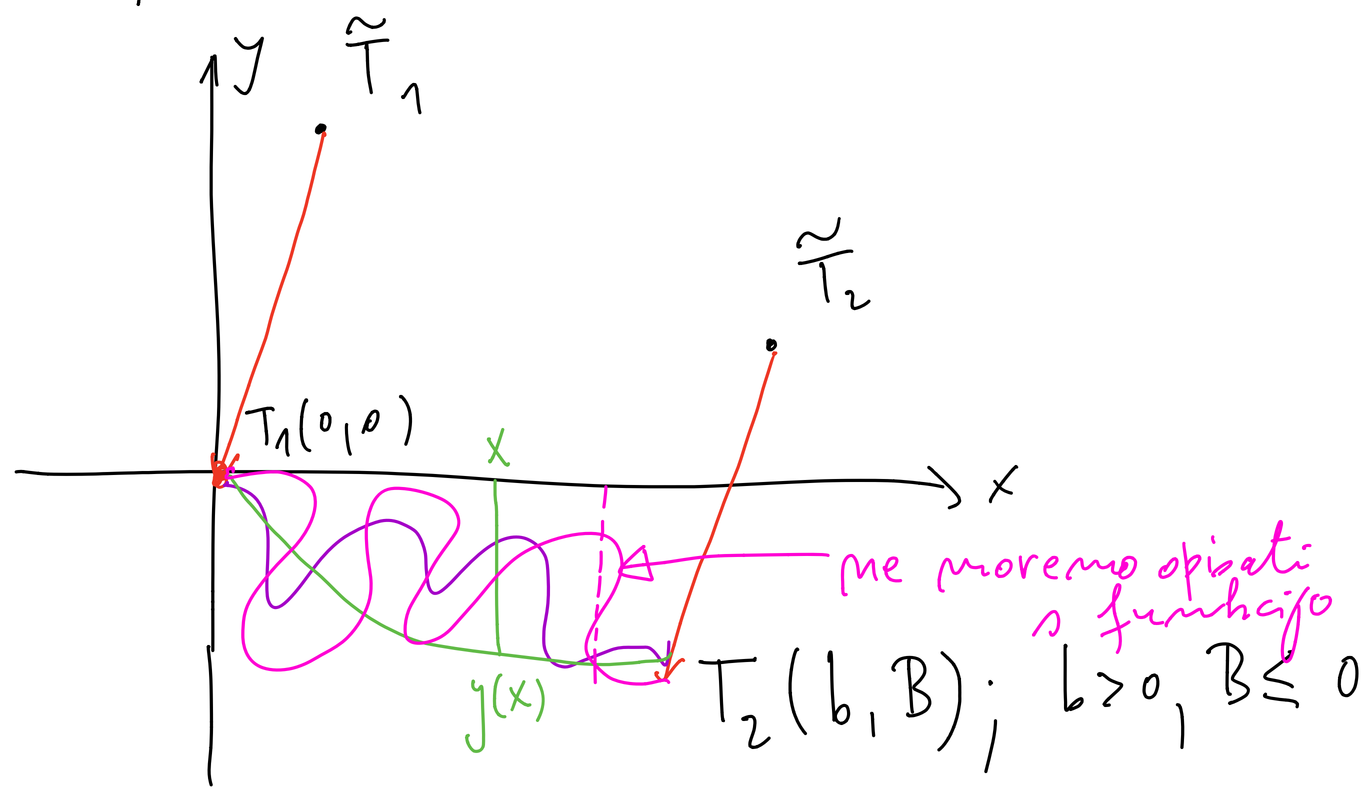
Problem brahistohrone



Iščemo krivuljo, po kateri bo kroglica najhitreje prišla iz T_1 v T_2 pod vplivom sile teže brez trenja.

1696 problem zastavi Johann Bernoulli

Čas, ki ga potrebuje kroglica je

$$t_{12} = \int_{T_1}^{T_2} \frac{ds}{v}$$

$$v = \frac{ds}{dt}$$

ds je element ločne dolžine:

$$ds = \sqrt{1+y'^2}\,dx$$

v: hitrost kroglice na višini y:

$$\frac{1}{2} m v^2 = mg(-y) \Rightarrow$$

$$\boxed{v = \sqrt{2g(-y)}}$$

$$t_{12} = \int_{T_1}^{T_2} \frac{\sqrt{1+y'^2}}{\sqrt{2g(-y)}}\,dx$$

$$= \int_0^1 \sqrt{\frac{1+y'(x)^2}{-2gy(x)}}\,dx \quad ; \quad y'(x) = \frac{dy}{dx}$$

Iščemo tako funkcijo y, da bo ta integral minimalen!

Če označimo $f(x,y,y') = \sqrt{\frac{1+y'^2}{-2gy}}$, potem iz variacijskega računa

sledi, da bo zgornji integral min.,

če bo veljalo:

$$\boxed{f(x,y,y') - y'\frac{\partial f}{\partial y'}(x,y,y') = k_1}$$

Euler - Lagrangeova enačba
(f ni odvisna eksplicitno od x)

Dobimo:

$$\sqrt{\frac{1+y'^2}{-2gy}} - y'\frac{\cancel{2}y'}{\sqrt{-2gy}\,\cancel{2}\sqrt{1+y'^2}} = k_1,$$

kar se preoblikuje (po množenju z

$\sqrt{-2gy}\sqrt{1+y'^2}$

$$1+\cancel{y'^2} - \cancel{y'^2} = k_1\sqrt{-2gy}\sqrt{1+y'^2} \Big/^2$$

$$\boxed{-y(1+y'^2) = k^2}$$ dif. enačba

$$k = \frac{1}{\sqrt{2g k_1^2}}$$

Kako rešujemo $-y(1+y'^2) = k^2$?

Uporabimo nastavek: $y = -k^2 \sin^2 t$:

$$\left(\frac{dy}{dx}\right)^2 = -\frac{k^2}{y} - 1 = \left(\frac{\cos t}{\sin t}\right)^2$$

$$\Rightarrow y' = \pm\sqrt{\left(\frac{\cos t}{\sin t}\right)^2}$$

$$= \pm \frac{\cos t}{\sin t} = -\frac{\cos t}{\sin t}$$

Iz zveze $dx = \frac{dx}{dy}\frac{dy}{dt}\,dt$,

z upoštevanjem nastavka in izpeljane relacije $y' = \frac{dy}{dx} = -\frac{\cos t}{\sin t}$, pridemo

do

$$dx = \cancel{-}\,\frac{\sin t}{\cancel{\cos t}}\left(\cancel{-}\,k^2\, 2 \sin t \cancel{\cos t}\right) dt$$

$$dx = 2k^2 \sin^2 t \, dt .$$

Od tod sledi

$$x = \int 2k^2 \sin^2 t \, dt$$

$$= \cancel{2}k^2 \int \frac{1-\cos 2t}{\cancel{2}} \, dt$$

$$= k^2\left(t - \frac{1}{2}\sin 2t\right) + C$$

Pri $t=0$, smo v točki $T_1(0,0)$, torej mora biti $x(0) = 0 \Rightarrow \underline{\underline{x(0) = 0 = C}}$.

$$x(t) = k^2\left(t - \frac{\sin 2t}{2}\right).$$

Uvedemo novo spremenljivko $\theta = 2t$

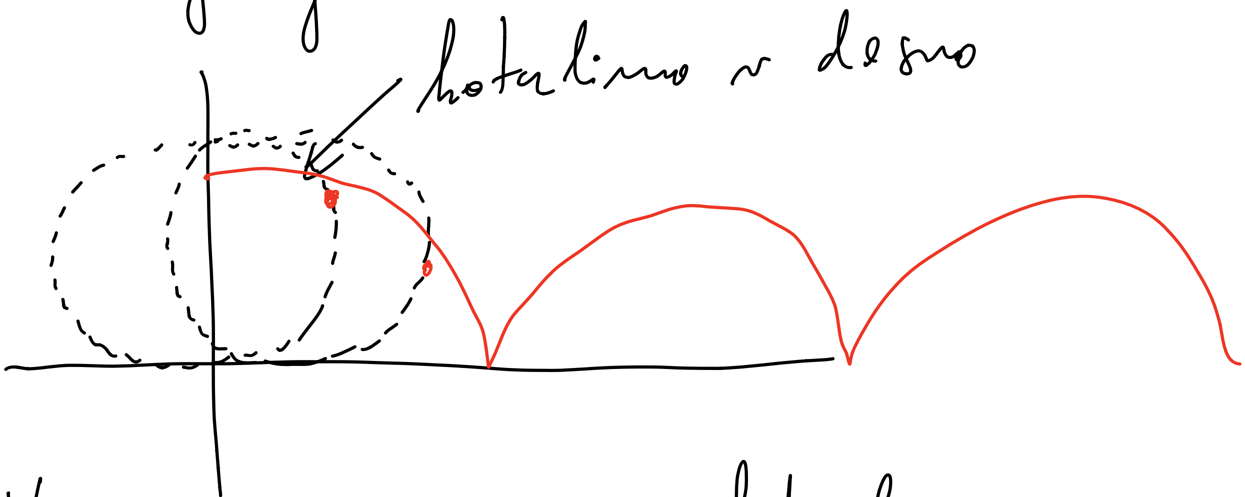
$$x(\theta) = k^2\left(\frac{\theta}{2} - \frac{\sin\theta}{2}\right) = \underline{\underline{\frac{k^2}{2}(\theta - \sin\theta)}}$$

$$y = -k^2 \sin^2 t = -k^2 \, \frac{1-\cos 2t}{2}$$

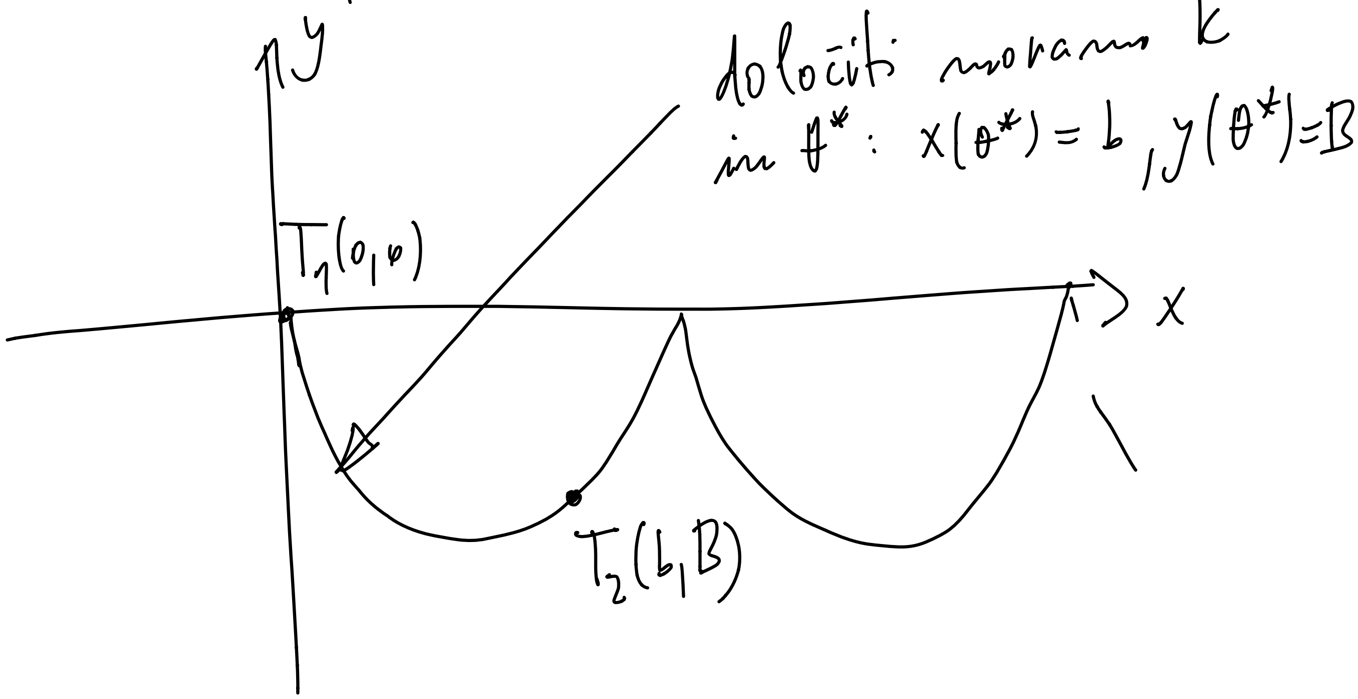
$$= -\frac{k^2}{2}\left(1-\cos\theta\right)$$

Dobili smo parametrični zapis krivulje: $x(\theta)$, $y(\theta)$, $\theta \geq 0$.

Krivulja je cikloida:



V našem primeru kotalimo



Rešiti moramo sistem enačb

$$x(\theta) = \frac{k^2}{2}(\theta - \sin\theta) = b > 0$$

$$y(\theta) = -\frac{k^2}{2}(1 - \cos\theta) = B \leq 0$$

Ko je $\theta = 0 \Rightarrow x = 0$ in $y = 0$! ✓

Če enačbi delimo in rezultat preuredimo,

dobimo

$$g(\theta) = 1 - \cos\theta + \frac{B}{b}(\theta - \sin\theta) = 0!$$

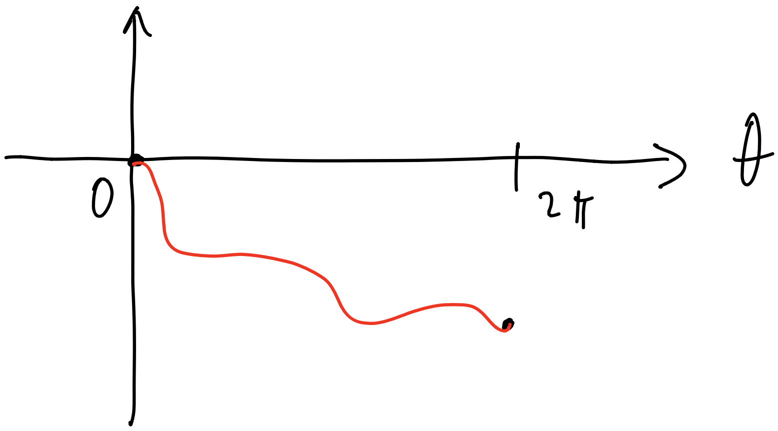
Rešiti moramo nelinearno enačbo

$$g(\theta) = 0!$$

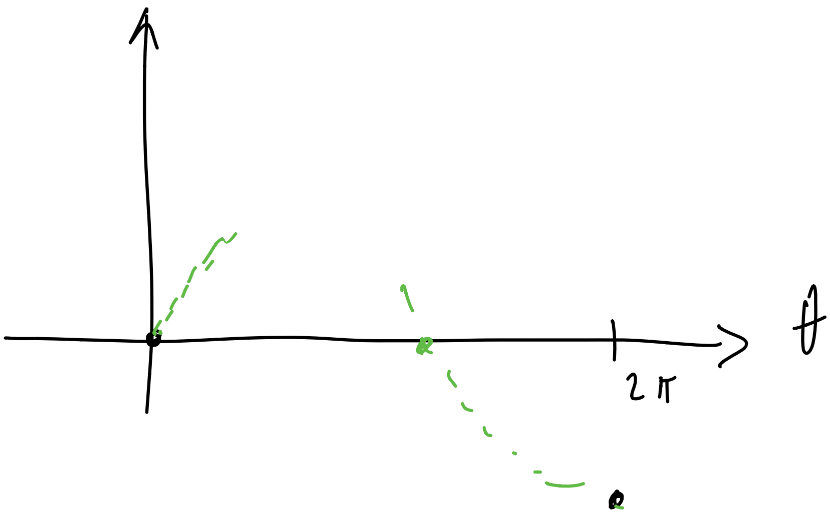
Poglejmo si obnašanje g na $[0, 2\pi]$!

<u>Lema</u>: Funkcija g ima na $(0, 2\pi)$ <u>natanko eno ničlo</u>!

<u>Dokaz</u>: $g(0) = 0$, $g(2\pi) = \frac{2\pi B}{b} < 0$



NI NIČLE



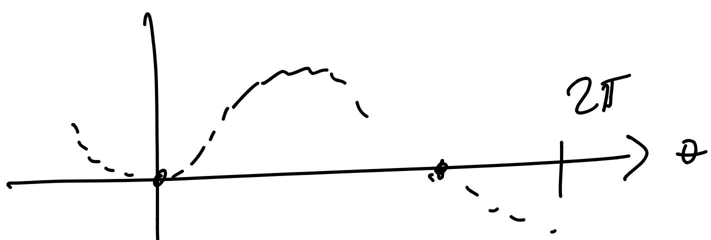
Dovolj je videti, da g narašča pri $\theta=0$:

$$g'(\theta) = \sin\theta + \frac{B}{b}(1-\cos\theta)$$

$$g'(0) = 0$$

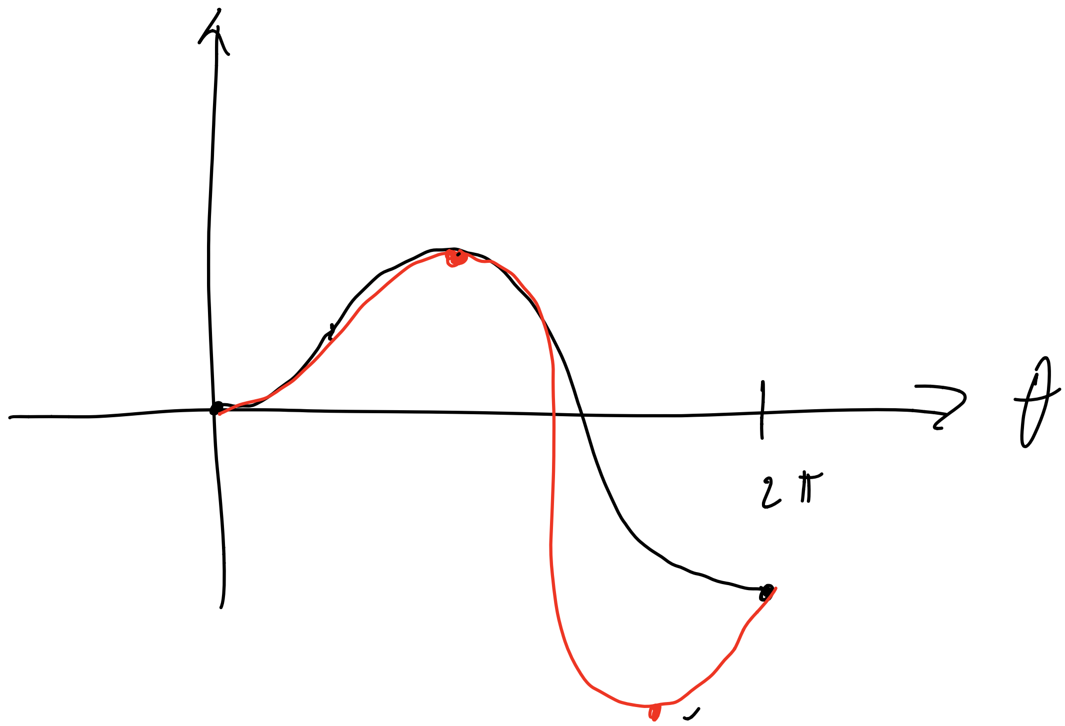
$$g''(\theta) = \cos\theta + \frac{B}{b}\sin\theta$$

$$g''(0) = 1 > 0 \Rightarrow \text{v } \theta=0 \text{ je minimum}$$



$\Rightarrow$ imamo vsaj eno ničlo na $(0, 2\pi)$.

Dokazati moramo, da je ničla ena sama!



Dovolj je dokazati, da ima g na $(0, 2\pi)$ natanko en maksimum!