INTEGRAL S PARAMETROM

1.12.2020

$$F(x)=\int_{u(x)}^{v(x)} f(x,t)\,dt \qquad x\in D\subset \mathbb{R}$$

u,v … funkciji ene spremenljivke f … funkcija dveh spremenljivk

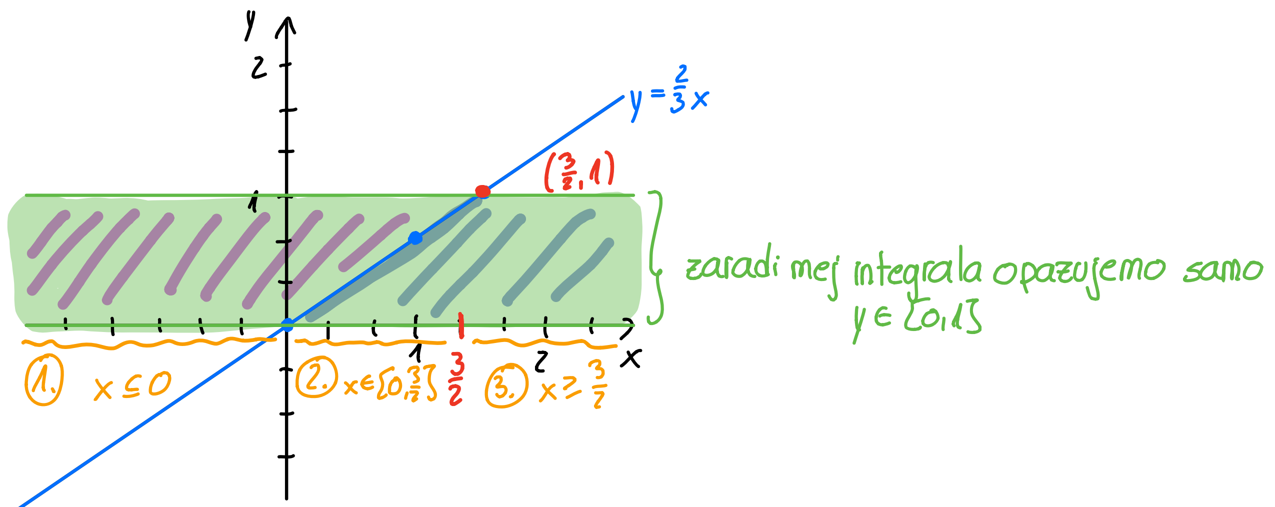
- u,v,f zvezne : F zvezna
- u,v,f zvezno odvedljive po x : $F'(x)=\int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(x,t)\,dt + v'(x)f(x,v(x)) - u'(x)f(x,u(x))$
- f zvezna na $[c,d]\times[a,b]$: $\int_c^d F(x)\,dx = \int_c^d \left(\int_a^b f(x,t)\,dt\right)dx = \int_a^b\left(\int_c^d f(x,t)\,dx\right)dt$

Naloga: Izračunaj $F(x)=\int_0^1 |2x-3y|\,dy$ za $x\in\mathbb{R}$.

Izraz $|2x-3y|$ ima na različnih območjih različne predpise.

$$|2x-3y| = \begin{cases} 2x-3y & ; \quad 2x-3y\geq 0 \quad \text{oz.} \quad y\leq \frac{2}{3}x \\ 3y-2x & ; \quad 2x-3y<0 \quad \text{oz.} \quad y>\frac{2}{3}x \end{cases}$$

Za katere vrednosti $x\in\mathbb{R}$ uporabljamo prvega in za katere drugega?



: $y\in[0,1]$, $y>\frac{2}{3}x$

: $y\in[0,1]$, $y\leq\frac{2}{3}x$

① : $x \leq 0$ in $y \in [0,1]$

Ker na tem območju velja $y > \frac{2}{3}x$, je $|2x-3y| = 3y-2x$

$$F(x) = \int_0^1 (3y-2x)\,dy = \left(3\cdot\frac{y^2}{2} - 2\cdot x\cdot y\right)\Big|_0^1 = \frac{3}{2} - 2x$$

pazi! integriramo po y

② : $x \in [0, \frac{3}{2}]$, $y \in [0,1]$

$$F(x) = \int_0^1 |3y-2x|\,dy = \int_0^{\frac{2}{3}x} |3y-2x|\,dy + \int_{\frac{2}{3}x}^1 |3y-2x|\,dy =$$

$0 \leq \frac{2}{3}x \leq 1$

$*\ \int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$

$$= \int_0^{\frac{2}{3}x} (2x-3y)\,dy + \int_{\frac{2}{3}x}^1 (3y-2x)\,dy =$$

$$= \left(2xy - 3\frac{y^2}{2}\right)\Big|_0^{\frac{2}{3}x} + \left(3\frac{y^2}{2} - 2xy\right)\Big|_{\frac{2}{3}x}^1 =$$

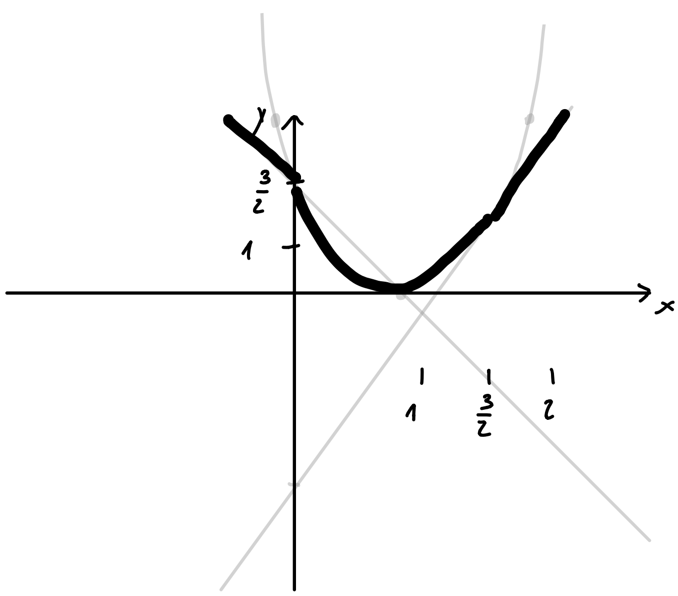
$$= \left(\frac{4}{3}x^2 - \frac{2}{3}x^2\right) - (0-0) + \left(\frac{3}{2} - 2x\right) - \left(\frac{2}{3}x^2 - \frac{4}{3}x^2\right) =$$

$$= 2\cdot\frac{4}{3}x^2 - 2\cdot\frac{2}{3}x^2 + \frac{3}{2} - 2x = \frac{4}{3}x^2 + \frac{3}{2} - 2x$$

③ : $x \in [\frac{3}{2}, \infty)$, $y \in [0,1]$ Na tem območju je $|2x-3y| = 2x-3y$.

$$F(x) = \int_0^1 2x-3y\,dy = \left(2xy - 3\frac{y^2}{2}\right)\Big|_0^1 = 2x - \frac{3}{2}$$

$$F(x) = \begin{cases} \frac{3}{2} - 2x & ; \quad x \leq 0 \\ \frac{4}{3}x^2 - 2x + \frac{3}{2} & ; \quad 0 \leq x \leq \frac{3}{2} \\ 2x - \frac{3}{2} & ; \quad \frac{3}{2} \leq x \end{cases}$$



Naloga: Izračunaj $\frac{d}{dx}\int_x^{\sqrt{x}} \frac{e^{-\frac{y^2}{x}}}{y}\,dy$.

$$F(x) := \int_x^{\sqrt{x}} \frac{e^{-\frac{y^2}{x}}}{y}\,dy$$

Ker je $\sqrt{x}$ definiran za $x \geq 0$ in je $\frac{y^2}{x}$ definiran za $x \neq 0$, je $F(x)$ definirana za $x > 0$.

- $u(x) = x$ zvezno odvedljiva po x

  $v(x) = \sqrt{x}$ zvezno odvedljiva po x

  $f(x) = \frac{e^{-\frac{y^2}{x}}}{y}$ zvezno odvedljiva po x

$$\Rightarrow F'(x) = \frac{d}{dx}\int_x^{\sqrt{x}} \frac{e^{-\frac{y^2}{x}}}{y}\,dy = \int_x^{\sqrt{x}} \frac{\partial}{\partial x}\frac{e^{-\frac{y^2}{x}}}{y}\,dy + \underbrace{\frac{1}{2}\frac{1}{\sqrt{x}}}_{v'(x)} \cdot \underbrace{\frac{e^{-\frac{\sqrt{x}^2}{x}}}{\sqrt{x}}}_{f(x,v(x))} - \underbrace{1}_{u'(x)} \cdot \underbrace{\frac{e^{-\frac{x^2}{x}}}{x}}_{f(x,u(x))}$$

$$F'(x) = \int_x^{\sqrt{x}} \frac{1}{\cancel{y}} \cdot \left(\frac{\cancel{y^2}}{x^2}\right) \cdot e^{-\frac{y^2}{x}}\,dy = \int_x^{\sqrt{x}} \frac{y}{x^2}\, e^{-\frac{y^2}{x}}\,dy + \frac{1}{2}\frac{e^{-1}}{x} - \frac{e^{-x}}{x} =$$

$$= \int_x^{\sqrt{x}} \frac{\cancel{y}}{\cancel{x^2}} \cdot e^t \cdot \left(-\frac{\cancel{x}}{2\cancel{y}}\right) dt + \frac{1}{2ex} - \frac{1}{xe^x} = -\int_{-x}^{-1} \frac{e^t}{2x}\,dt + \frac{1}{2ex} - \frac{1}{xe^x} =$$

uvedba nove spremenljivke

$t = -\frac{y^2}{x}$ (x je znotraj integrala kot konstanta)

$dt = -\frac{1}{x} \cdot 2y\,dy$

$dy = -\frac{x}{2y}\,dt$

Integralske meje:

$u(x) = x \rightarrow u(t) = -\frac{(\sqrt{x})^2}{x} = -1$

$v(x) = \sqrt{x} \rightarrow v(t) = -\frac{x^2}{x} = -x$

$$= -\frac{1}{2x} \cdot e^t \Big|_{-x}^{-1} + \frac{1}{2ex} - \frac{1}{xe^x} = -\frac{1}{2x} \cdot \left(\cancel{\frac{1}{e}} - \frac{1}{e^x}\right) + \frac{1}{2x} \cdot \left(\cancel{\frac{1}{e}} - \frac{2}{e^x}\right) =$$

$$= \frac{1}{2x} \cdot \left(\frac{1}{e^x} - \frac{2}{e^x}\right) = \underline{\underline{-\frac{1}{2x} \cdot \frac{1}{e^x}}}$$

Naloga: Naj bo $F(y) = \int_{2y}^{y^2} \sin(xy)\,dx$. Izračunaj $F'(y)$.

$$F'(y) = \int_{2y}^{y^2} \frac{\partial}{\partial y} \sin(x\cdot y)\,dx + (y^2)'\cdot \sin(y^2\cdot y) - (2y)'\cdot \sin(2y\cdot y) =$$

$$= \int_{2y}^{y^2} \cos(x\cdot y)\cdot x\,dx + 2y\cdot \sin y^3 - 2\cdot \sin(2y^2) =$$

$u = x$
$du = \cos(xy)dx$

$du = 1$
$v = \sin(xy)\cdot \frac{1}{y}$

$$\overset{\text{P.P.}}{=} x\cdot \sin(xy)\cdot \frac{1}{y}\Big|_{x=2y}^{y^2} - \int_{2y}^{y^2} \sin(xy)\cdot \frac{1}{y}\,dx + 2y\cdot \sin y^3 - 2\cdot \sin(2y^2) =$$

$$= \frac{y^{\cancel{2}}}{\cancel{y}}\cdot \sin(y^2\cdot y) - (2\cancel{y})\cdot \sin(2y\cdot y)\cdot \frac{1}{\cancel{y}} - \frac{1}{y}\cdot \left(-\frac{\cos(xy)}{y}\right)\Big|_{y=2y}^{y^2} + 2y\cdot \sin y^3 - 2\cdot \sin(2y^2) =$$

$$= y\cdot \sin y^3 - 2\cdot \sin(2y^2) + \frac{1}{y^2}\cdot \left(\cos y^3 - \cos(2y^2)\right) + 2y\cdot \sin y^3 - 2\cdot \sin(2y^2) =$$

$$= 3y\cdot \sin y^3 - 4\sin(2y^2) + \frac{1}{y^2}\cos y^3 - \frac{1}{y^2}\cos(2y^2)$$

Druga možnost: $F(y) = -\frac{\cos(xy)}{y}\Big|_{x=2y}^{y^2} = -\frac{\cos(y^3)}{y} + \frac{\cos(2y^2)}{y}$ $\qquad F'(y) = \ldots$ (DN!)

Naloga: Dokaži, da je integral

$$\int_{e^{\frac{1}{x+1}}}^{e^{\frac{2}{x+1}}} \frac{y^x}{\ln y}\,dy$$

neodvisen od x.

Namig: Najprej pokaži, da to velja na poltrakih $(-\infty,-1)$ in $(-1,\infty)$.
Nato pa s substitucijo $z = \frac{1}{y}$ pokaži še, da se vrednost na prvem poltraku ujema z vrednostjo na drugem poltraku.

$$F(x) := \int_{e^{\frac{1}{x+1}}}^{e^{\frac{2}{x+1}}} \frac{y^x}{\ln y}\,dy$$

$F(x)$ je definirana za $x \in \mathbb{R}\setminus\{-1\}$

$F(x)$ neodvisen od x, če $F'(x) = 0$.

Izračunajmo $F'(x)$:

$$F'(x) = \int_{e^{\frac{1}{x+1}}}^{e^{\frac{2}{x+1}}} \frac{\partial}{\partial x} \frac{y^x}{\ln y}\, dy + \left(e^{\frac{2}{x+1}}\right)' \cdot \frac{\left(e^{\frac{2}{x+1}}\right)^x}{\ln e^{\frac{2}{x+1}}} - \left(e^{\frac{1}{x+1}}\right)' \cdot \frac{\left(e^{\frac{1}{x+1}}\right)^x}{\ln e^{\frac{1}{x+1}}} =$$

za y vstavimo $e^{\frac{2}{x+1}}$

$$= \int_{e^{\frac{1}{x+1}}}^{e^{\frac{2}{x+1}}} \frac{1}{\cancel{\ln y}} \cdot y^x \cdot \cancel{\ln y}\, dy + e^{\frac{2}{x+1}} \cdot (-\cancel{2}) \frac{1}{(x+1)^{\cancel{2}}} \cdot \frac{e^{\frac{2x}{x+1}}}{\frac{\cancel{2}}{\cancel{x+1}}} - e^{\frac{1}{x+1}} \cdot (-1) \frac{1}{(x+1)^{\cancel{2}}} \cdot \frac{e^{\frac{x}{x+1}}}{\frac{1}{\cancel{x+1}}} =$$

$$= \frac{y^{x+1}}{x+1}\Bigg|_{e^{\frac{1}{x+1}}}^{e^{\frac{2}{x+1}}} - e^{\frac{2}{x+1}} \cdot e^{\frac{2x}{x+1}} \cdot \frac{1}{x+1} + e^{\frac{1}{x+1}} \cdot e^{\frac{x}{x+1}} \cdot \frac{1}{x+1} =$$

$$= \frac{1}{x+1}\left(\left(e^{\frac{2}{\cancel{x+1}}}\right)^{\cancel{x+1}} - \left(e^{\frac{1}{\cancel{x+1}}}\right)^{\cancel{x+1}}\right) - e^{\frac{2+2x}{x+1}} \cdot \frac{1}{x+1} + e^{\frac{1+x}{x+1}} \cdot \frac{1}{x+1} =$$

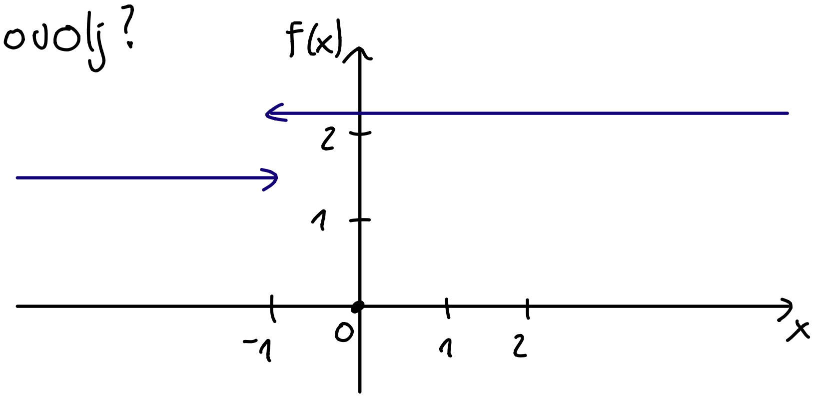
$$= \frac{1}{x+1}\left(e^2 - e^1\right) - e^2 \cdot \frac{1}{x+1} + e^1 \cdot \frac{1}{x+1} =$$

$$= \frac{1}{x+1}\left(\cancel{e^2} - \cancel{e} - \cancel{e^2} + \cancel{e}\right) = \underline{\underline{0}}$$

Pokazali smo, da je $F'(x) = 0$ za $x \neq 0$ $\Rightarrow$

$F(x)$ je konstantna na intervalih zveznosti, torej na $(-\infty, -1)$ in $(-1, \infty)$.

Zakaj to ni dovolj?



lahko bi se zgodilo kaj takega

Pokažimo torej, da $F(x)$ na obeh intervalih zavzame isto vrednost.

Upoštevamo namig in upeljemo novo spremenljivko $z = \frac{1}{y} = y^{-1} \Rightarrow y = \frac{1}{z}$

$$dy = -\frac{1}{z^2}\,dz$$

$$u(x) = e^{\frac{1}{x+1}} \longrightarrow u(z) = e^{-\frac{1}{x+1}}$$

$$v(x) = e^{\frac{2}{x+1}} \longrightarrow v(z) = e^{-\frac{2}{x+1}}$$

$$F(x) := \int_{e^{\frac{1}{x+1}}}^{e^{\frac{2}{x+1}}} \frac{y^x}{\ln y}\,dy = \int_{e^{-\frac{1}{x+1}}}^{e^{-\frac{2}{x+1}}} \frac{z^{-x}}{\ln z^{-1}} \left(-\frac{1}{z^2}\right) dz =$$

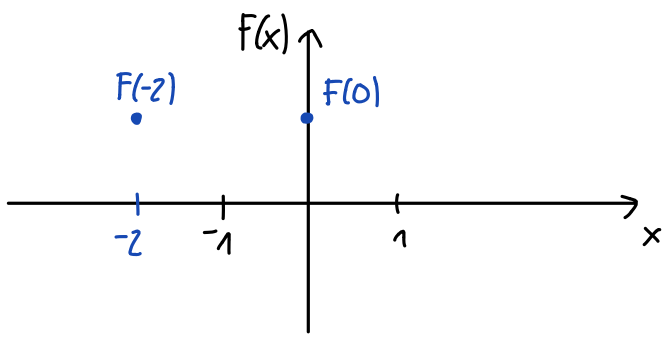
$$-(x+1) = (-x-2)+1$$

$$= \int_{e^{-\frac{1}{x+1}}}^{e^{-\frac{2}{x+1}}} \frac{z^{-x}}{\cancel{-}\ln z} \left(\cancel{-}\frac{1}{z^2}\right) dz = \int_{e^{-\frac{1}{x+1}}}^{e^{-\frac{2}{x+1}}} \frac{z^{-x-2}}{\ln z}\,dz = \int_{e^{\frac{1}{(-x-2)+1}}}^{e^{\frac{2}{(-x-2)+1}}} \frac{z^{-x-2}}{\ln z}\,dz =$$

$$= \int_{e^{\frac{1}{(-x-2)+1}}}^{e^{\frac{2}{(-x-2)+1}}} \frac{y^{-x-2}}{\ln y}\,dy = \int_{e^{\frac{1}{t+1}}}^{e^{\frac{2}{t+1}}} \frac{y^t}{\ln y}\,dy = F(t) \overset{t=-x-2}{=} F(-x-2)$$

$t=-x-2$ definicija $F(x)$

Ugotovili smo: $\boxed{F(x) = F(-x-2)}$!



- Ker so vrednosti $F(x)$ na $(-\infty,-1)$ konstantne, so vse enake $F(-2)$.
- -II- na $(-1,\infty)$ konstantne, -II- $F(0)$

$F(-2) = F(-(-2)-2) = F(2-2) = F(0)$

- Ker sta vrednosti $F(0)$ in $F(-2)$ enaki, je F konstantna na $\mathbb{R}\setminus\{-1\}$.

Naloga: Izračunaj $g'(x)$, kjer je $g(x)=\int_0^{\frac{\pi}{2}} \frac{\sin(xy)}{y}\,dy$.

8.12.2020

- $u(x)=0$, $v(x)=\frac{\pi}{2}$: zvezno odvedljivi ✓
- $f(x,y)=\frac{\sin(xy)}{y}$ zvezna?

$$\frac{\sin(xy)}{y}=x\cdot\frac{\sin(xy)}{xy} \qquad \lim_{y\to 0} x\cdot\underbrace{\frac{\sin(\cancel{x}y)}{\cancel{x}\cdot y}}_{\to 1}=x$$

$\Rightarrow$ f lahko razširimo do zvezne funkcije na $\mathbb{R}\times[0,\frac{\pi}{2}]$

$$f(x,y)=\begin{cases}\frac{\sin(xy)}{y} & ;\ y\neq 0\\ x & ;\ y=0\end{cases} \qquad (\text{zvezna na } \mathbb{R}\times\mathbb{R})$$

$\Rightarrow$ g zvezna na $\mathbb{R}$ (glej lastnosti na začetku)

$$\frac{\partial f}{\partial x}(x,y)=\frac{1}{\cancel{y}}\cdot\cos(xy)\cdot\cancel{y}=\cos(xy) \leftarrow \text{zvezna na } \mathbb{R}\times[0,\tfrac{\pi}{2}]$$

$\Rightarrow$ f zvezno odvedljiva po x

vrednosti za y!

$$\Rightarrow g'(x)=\int_0^{\frac{\pi}{2}}\frac{\partial}{\partial x}f(x,y)\,dy=\int_0^{\frac{\pi}{2}}\cos(xy)\,dy=\frac{\sin(x\cdot y)}{x}\Big|_0^{\frac{\pi}{2}}=\frac{\sin(\frac{\pi}{2}\cdot x)}{x}-\frac{\sin(0\cdot x)}{x}=$$

$$=\frac{\sin(\frac{\pi}{2}\cdot x)}{x} \quad \text{če } x\neq 0$$

Če je $x=0$: $g'(x)=\int_0^{\frac{\pi}{2}}\cos(0\cdot y)\,dy=\int_0^{\frac{\pi}{2}}dy=\frac{\pi}{2}-0=\frac{\pi}{2}$

$$\Rightarrow g'(x)=\begin{cases}\frac{\sin(\frac{\pi}{2}\cdot x)}{x} & ;\ x\neq 0\\ \frac{\pi}{2} & ;\ x=0\end{cases}$$

Naloga: a) Za $\eta \geq 0$ izračunaj $F(\eta) = \int_0^\eta \frac{\ln(1+\eta x)}{1+x^2}\,dx$.

b) Izračunaj $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$.

DODATNO!

(nismo rešili na vajah)

a) $f(\eta, x) = \frac{\ln(1+\eta x)}{1+x^2}$

za $0 \leq x \leq \eta \Rightarrow 1+\eta x \geq 1$ in $1+x^2 \geq 1$

$\Rightarrow f$ je definirana in zvezna za $\eta \geq 0$

$\Rightarrow F(\eta) = \int_0^\eta \frac{\ln(1+\eta x)}{1+x^2}\,dx$ je zvezna na $[0, \infty)$

$\frac{\partial f}{\partial \eta} = \frac{x}{(1+\eta x)\cdot(1+x^2)}$ je definiran in zvezna za $\eta > 0$ in $x \geq 0$

$$\Rightarrow F'(\eta) = \int_0^\eta \frac{x}{(1+\eta x)\cdot(1+x^2)}\,dx + (\eta)' \cdot f(\eta, \eta) - (0)' \cdot f(\eta, 0) =$$

$$= \int_0^\eta \frac{x}{(1+\eta x)\cdot(1+x^2)}\,dx + 1 \cdot \frac{\ln(1+\eta^2)}{1+\eta^2} - 0 = *$$

$$\frac{x}{(1+\eta x)\cdot(1+x^2)} = \frac{A}{1+\eta x} + \frac{Bx+C}{1+x^2} = \frac{A + Ax^2 + Bx + C + B\eta x^2 + C\eta x}{(1+\eta x)(1+x^2)}$$

x^0: $0 = A + C \rightarrow C = -A$

x^1: $1 = B + C\eta \rightarrow 1 = B - A\eta \rightarrow 1 = B + B\eta^2$

$1 = B \cdot (1+\eta^2) \rightarrow \boxed{B = \frac{1}{1+\eta^2}}$

x^2: $0 = A + B\eta \rightarrow A = -B\eta$

$\boxed{A = \frac{-\eta}{1+\eta^2}}$ $\rightarrow$ $\boxed{C = \frac{\eta}{1+\eta^2}}$

$$* = \int_0^\eta -\frac{\eta}{1+\eta^2} \cdot \frac{1}{1+\eta x}\,dx + \int_0^\eta \frac{1}{1+\eta^2} \cdot \frac{x}{1+x^2}\,dx + \int_0^\eta \frac{\eta}{1+\eta^2} \cdot \frac{1}{1+x^2} + \frac{\ln(1+\eta^2)}{1+\eta^2} =$$

$$= \frac{1}{1+\eta^2} \cdot \left(-\eta \cdot \int_0^\eta \frac{1}{1+\eta x}\,dx + \int_0^\eta \frac{x}{1+x^2}\,dx + \eta \cdot \int_0^\eta \frac{1}{1+x^2}\,dx + \ln(1+\eta^2) \right) =$$

$t = 1+x^2$

$dt = 2x\,dx$

$$= \frac{1}{1+\eta^2} \cdot \left(-\cancel{\eta} \ln|1+\eta x| \cdot \frac{1}{\cancel{\eta}} \Big|_{x=0}^{\eta} + \int_1^{1+\eta^2} \frac{\cancel{x}}{t} \cdot \frac{dt}{2\cancel{x}} + \eta \cdot \arctan x \Big|_{x=0}^{\eta} + \ln(1+\eta^2) \right) =$$

$$= \frac{1}{1+\eta^2} \cdot \left(-\cancel{\ln(1+\eta^2)} + 0 + \frac{1}{2} \cdot \ln|t| \Big|_{t=1}^{1+\eta^2} + \eta \arctan\eta - \eta \arctan 0 + \cancel{\ln(1+\eta^2)} \right) =$$

$$= \frac{1}{1+\eta^2} \cdot \left(\frac{1}{2} \cdot \ln(1+\eta^2) - \frac{1}{2} \cdot 0 + \eta \arctan\eta - 0 \right) = \frac{1}{1+\eta^2} \cdot \left(\frac{1}{2} \ln(1+\eta^2) + \eta \cdot \arctan\eta \right)$$

$$\Rightarrow F'(\alpha) = \frac{1}{1+\alpha^2} \cdot \left(\frac{1}{2} \cdot \ln(1+\alpha^2) + \alpha \cdot \arctan\alpha \right)$$

$$\Rightarrow F(\alpha) = \int F'(\alpha)\, d\alpha = \int \frac{1}{1+\alpha^2} \cdot \left(\frac{1}{2} \cdot \ln(1+\alpha^2) + \alpha \cdot \arctan\alpha \right) d\alpha =$$

$$= \underbrace{\frac{1}{2} \int \frac{1}{(1+\alpha^2)} \cdot \ln(1+\alpha^2)\, d\alpha}_{} + \int \frac{1}{1+\alpha^2} \cdot \alpha \cdot \arctan\alpha\, d\alpha =$$

$u = \ln(1+\alpha^2) \qquad dv = \frac{1}{1+\alpha^2}\, d\alpha$

$du = \frac{2\alpha}{1+\alpha^2}\, d\alpha \qquad v = \arctan\alpha$

$$= \frac{1}{2} \cdot \left(\ln(1+\alpha^2) \cdot \arctan\alpha - \int \frac{2\alpha}{1+\alpha^2} \cdot \arctan\alpha\, d\alpha \right) + \int \frac{\alpha}{1+\alpha^2} \cdot \arctan\alpha\, d\alpha =$$

$$= \frac{1}{2} \ln(1+\alpha^2) \cdot \arctan\alpha \cancel{- \int \frac{\alpha}{1+\alpha^2} \arctan\alpha\, d\alpha} \cancel{+ \int \frac{\alpha}{1+\alpha^2} \arctan\alpha\, d\alpha} =$$

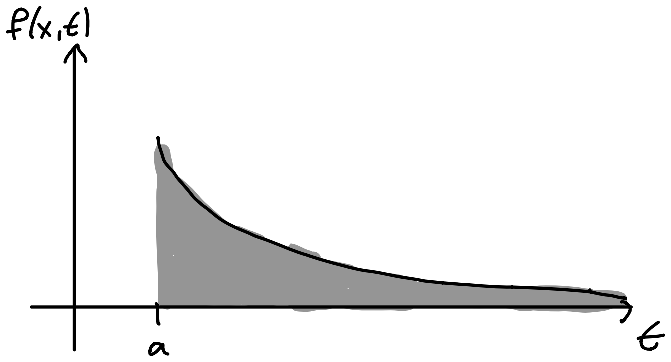
$$= \frac{1}{2} \cdot \ln(1+\alpha^2) \cdot \arctan\alpha + C$$

$$\left.\begin{aligned} &\Rightarrow F(\alpha) = \frac{1}{2} \cdot \ln(1+\alpha^2) \cdot \arctan\alpha + C \quad \Rightarrow F(0) = \frac{1}{2} \cdot \ln 1 \cdot \arctan 0 + C = C \\ &F(\alpha) = \int_0^\alpha \frac{\ln(1+\alpha x)}{1+x^2}\, dx \quad \Rightarrow F(0) = \int_0^0 \frac{\ln 1}{1+x^2}\, dx = 0 \end{aligned}\right\} C = 0$$

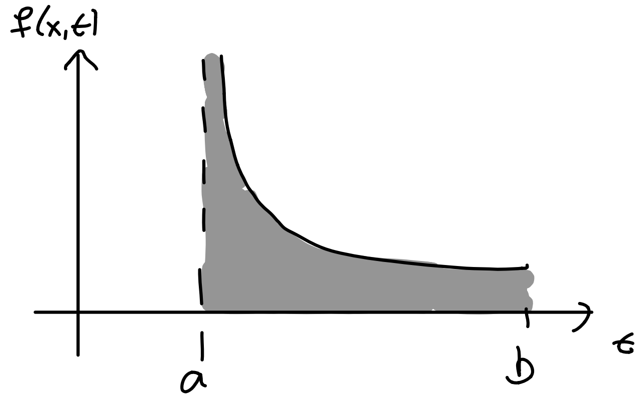
$$\Rightarrow F(\alpha) = \frac{1}{2} \cdot \ln(1+\alpha^2) \cdot \arctan\alpha$$

b) $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx = F(1) = \frac{1}{2} \cdot \ln(1+1^2) \cdot \arctan 1 = \frac{1}{2} \cdot \ln 2 \cdot \frac{\pi}{4} = \underline{\underline{\frac{\pi}{8} \cdot \ln 2}}$

POSPLOŠENI INTEGRAL



$$F(x)=\int_a^\infty f(x,t)\,dt$$



$$F(x)=\int_a^b f(x,t)\,dt$$

$x\in I$

<u>Naloga</u>: Določi definicijsko območje integrala $\int_0^\infty e^{a\sqrt{y}}\,dy$ s parametrom a.

$a>0$: $$\int_0^\infty e^{a\sqrt{y}}\,dy = \int_0^\infty \frac{2}{a^2} z e^z\,dz = \frac{2}{a^2}\int z e^z\,dz = \frac{2}{a^2}\left(z e^z\Big|_0^\infty - \int_0^\infty e^z\,dz \right) =$$

$z=a\sqrt{y}$

$dz = a\frac{1}{2}\frac{1}{\sqrt{y}}\,dy$

$\frac{1}{a^2}2z\,dz = dy$

$u=z$ $\quad dv=e^z dz$

$du=dz$ $\quad v=e^z$

$$=\frac{2}{a^2}\left(z e^z\Big|_0^\infty - e^z\Big|_0^\infty \right) = \frac{2}{a^2}\left(\lim_{z\to\infty}\left(z e^z - e^z\right) - 0\cdot e^0 + e^0 \right) =$$

$$=\frac{2}{a^2}\lim_{z\to\infty} e^z\cdot(z-1) + 1 = \infty$$

$\Rightarrow$ za $a>0$ ta integral ne obstaja

$a=0$: $$\int_0^\infty e^0\,dy = \int_0^\infty 1\,dy = y\Big|_0^\infty = \infty$$ $\Rightarrow$ za $a=0$ integral ne obstaja

$a<0$: $$\int_0^\infty e^{a\sqrt{y}}\,dy = \frac{2}{a^2}\int_0^{-\infty} z e^z\,dz = \frac{2}{a^2}\left(z e^z\Big|_0^{-\infty} - \int_0^{-\infty} e^z\,dz \right) = \frac{2}{a^2}\left(z e^z\Big|_0^{-\infty} - e^z\Big|_0^{-\infty} \right) =$$

$z=a\sqrt{y}$

$$=\frac{2}{a^2}\left(\lim_{z\to-\infty} e^z(z-1) - 0\cdot e^0 + e^0 \right) =$$

$$=\frac{2}{a^2}\left(\lim_{w\to\infty} \frac{-w-1}{e^w} + 1 \right) = \frac{2}{a^2}\cdot(0+1) = \frac{2}{a^2}$$

Naloga: Določi definicijsko območje integrala $\int_0^\infty \frac{\ln(1+ay^2)}{y^2}dy$ s parametrom a.

$$F(a)=\int_0^\infty \frac{\ln(1+ay^2)}{y^2}dy$$

če $a\geq 0$ potem $1+ay^2\geq 1 \Rightarrow \ln(1+ay^2)$ je definiran

$a<0$: $1+ay^2<0$ od nekega y naprej $(y\in(0,\infty)) \Rightarrow$ za $a<0$ integral ni definiran

$a=0$: $F(a)=F(0)=\int_0^\infty \frac{\ln 1}{y^2}dy=0$

$a>0$: $F(a)=\int_0^\infty \frac{\ln(1+ay^2)}{y^2}dy=-\frac{1}{y}\cdot\ln(1+ay^2)\Big|_{y=0}^{\infty}-\int_0^\infty(-\cancel{\tfrac{1}{y}})\cdot\frac{1}{1+ay^2}\cdot 2a\cancel{y}\,dy=$

$u=\ln(1+ay^2)$ $\quad dv=\frac{1}{y^2}dy$

$du=\frac{1}{1+ay^2}\cdot 2ay$ $\quad v=-\frac{1}{y}$

$$=-\lim_{y\to\infty}\frac{\ln(1+ay^2)}{y}+\lim_{y\to 0}\frac{\ln(1+ay^2)}{y}+2a\cdot\int_0^\infty\frac{1}{1+ay^2}dy=$$

$$=-\underbrace{\lim_{y\to\infty}\frac{\frac{2ay}{1+ay^2}}{1}}_{L.P.}+\underbrace{\lim_{y\to 0}\frac{\frac{2ay}{1+ay^2}}{1}}_{L.P.}+2a\cdot\int_0^\infty\frac{1}{1+(\sqrt{a}y)^2}dy=$$

$$=-\lim_{y\to\infty}\frac{2ay}{1+ay^2}+\lim_{y\to 0}\frac{2ay}{1+ay^2}+2a\cdot\arctan(\sqrt{a}\cdot y)\cdot\frac{1}{\sqrt{a}}\Big|_{y=0}^{\infty}=$$

$$=-\lim_{y\to\infty}\frac{\frac{2a}{y}}{\frac{1}{y^2}+a}+0+2\sqrt{a}\cdot\arctan(\sqrt{a}\cdot y)\Big|_{y=0}^{\infty}=$$

$$=-0+2\sqrt{a}\cdot\left(\frac{\pi}{2}-0\right)=\cancel{2}\sqrt{a}\cdot\frac{\pi}{\cancel{2}}=\underline{\underline{\pi\cdot\sqrt{a}}}$$

$\Rightarrow D_F=[0,\infty)$ ker je $F(a)=\pi\cdot\sqrt{a}$ za $a>0$, $F(0)=0$

in $F(a)$ ne obstaja za $a<0$.

Weistrassov kriterij za integrale s parametri:

Če $\exists\, g(t)$ zvezna, za katero velja $|f(x,t)| < g(t)$ za $\forall x \in I$ in $\forall t \in [a,\infty)$ oz. $[a,b]$

in integral $\int_a^\infty g(t)\,dt$ oz. $\int_a^b g(t)\,dt$ konvergira

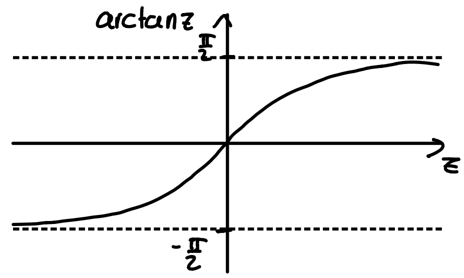
$\Rightarrow \int_a^\infty f(x,t)\,dt$ oz. $\int_a^b f(x,t)\,dt$ konvergira enakomerno

Zveznost integrala s parametrom

$f(x,t)$ zvezna, $F(x)=\int_a^\infty f(x,t)\,dt$, F konvergira enakomerno $\Rightarrow F$ zvezna

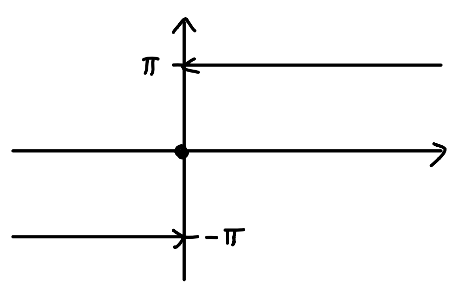
Naloga: Ali je $F(x)=\int_{-\infty}^{\infty} \frac{x}{1+x^2y^2}\,dy$ zvezna? Kaj lahko iz tega sklepamo?

$$\frac{x}{1+\underbrace{x^2y^2}_{\geq 0}}$$ zvezna funkcija dveh spremenljivk



$$F(x)=\int_{-\infty}^{\infty}\frac{x}{1+x^2y^2}\,dy \underset{\substack{t=xy\\ dt=x\,dy}}{=} \begin{cases} \int_{-\infty}^{\infty}\frac{\cancel{x}}{1+t^2}\frac{dt}{\cancel{x}} & ; x>0 \\ \int_{-\infty}^{\infty}\frac{0}{1+0}\,dy & ; x=0 \\ \int_{\infty}^{-\infty}\frac{\cancel{x}}{1+t^2}\frac{dt}{\cancel{x}} & ; x<0 \end{cases} = \begin{cases} \arctan t\,\Big|_{-\infty}^{\infty} & ; x>0 \\ 0 & ; x=0 \\ \arctan t\,\Big|_{\infty}^{-\infty} & ; x<0 \end{cases} =$$

$$= \begin{cases} \frac{\pi}{2}-\left(-\frac{\pi}{2}\right) & ; x>0 \\ 0 & ; x=0 \\ -\frac{\pi}{2}-\frac{\pi}{2} & ; x<0 \end{cases} = \begin{cases} \pi & ; x>0 \\ 0 & ; x=0 \\ -\pi & ; x<0 \end{cases}$$



$f(x,y)$ zvezna funkcija $\not\Rightarrow$ $F(x)$ zvezna

$\Rightarrow F$ ne konvergira enakomerno

Odvajanje posplošenih integralov

$$F(x)=\int_a^\infty f(x,t)\,dt \quad \text{oz.} \quad F(x)=\int_a^b f(x,t)\,dt \qquad \text{za } x\in I,\ I \text{ odprt interval}$$

Če

- $F(x)$ obstaja za neki $x\in I$
- f je parcialno odvedljiva po x
- parcialni odvod f_x je zvezen v obeh spremenljivkah
- $\int_a^\infty \frac{\partial}{\partial x} f(x,t)\,dt$ oz. $\int_a^b \frac{\partial}{\partial x} f(x,t)\,dt$ je enakomerno konvergenten: po Weistrassovem kriteriju:

  obstaja tak $g(t)$, da $\left|\frac{\partial}{\partial x} f(x,t)\right|\le g(t)$ za $\forall x\in I, t\in[a,\infty)$ oz. $t\in[a,b)$

  in $\int_a^\infty g(t)\,dt$ oz. $\int_a^b g(t)\,dt$ konvergira

Potem:
- $F(x)$ obstaja za $\forall x\in I$
- $F'(x)=\int_a^\infty f_x(x,t)\,dt$ oz. $F'(x)=\int_a^b f_x(x,t)\,dt$

Naloga: Izračunaj $\int_0^\infty \frac{1-e^{-ax^2}}{xe^{x^2}}\,dx$ za $a\ge 0$.

$$F(a)=\int_0^\infty \frac{1-e^{-ax^2}}{xe^{x^2}}\,dx \qquad \text{Pazi! parameter je } a!$$

- $F(0)=\int_0^\infty \frac{1-e^{-0}}{x\cdot e^{x^2}}\,dx=0 \Rightarrow F$ obstaja za nek $x\in I$.
- $f(a,x)=\frac{1-e^{-ax^2}}{x\cdot e^{x^2}}$ Ali obstaja f_a? ✓ $f_a(a,x)=\frac{e^{-ax^2}\cdot x^2}{x\cdot e^{x^2}}=x\cdot e^{-x^2(1+a)}$
- $f_a(a,x)=x\cdot e^{-x^2\cdot\underbrace{(1+a)}_{\ge 1}}\le x\cdot e^{-x^2}$ za $x\in[0,\infty)$ (ker $a\ge 0$)

$$\int_0^\infty xe^{-x^2}dx = \int_0^{-\infty} \cancel{x}e^{t}\cdot\left(-\frac{1}{2\cancel{x}}\right)dt=\frac{1}{2}\int_{-\infty}^0 e^t\,dt=\frac{1}{2}e^t\Big|_{-\infty}^0=\frac{1}{2}-0=\frac{1}{2}$$

$t=-x^2$
$dt=-2x$

$\Rightarrow \int_0^\infty g(t)\,dt$ obstaja $\Rightarrow \int_0^\infty xe^{-(1+a)x^2}dx$ enakomerno konvergira

$\Rightarrow$ odvod lahko računamo z odvajanjem pod integralom

$$\Rightarrow F'(a)=\int_0^\infty xe^{-(1+a)x^2}dx=-\frac{1}{1+a}\cdot\frac{1}{2}\int_0^{-\infty}\cancel{x}e^t\cdot\cancel{\frac{1}{x}}\,dt=\frac{1}{2(1+a)}e^t\Big|_{-\infty}^0=\underline{\underline{\frac{1}{2(1+a)}}}$$

$t=-(1+a)x^2$
$dt=-(1+a)2x\,dx$

Naloga: Izračunajte integral 15.12.2020

$$\int_0^\infty \frac{\arctan(xy)}{y(1+y^2)}\,dy \qquad (x\in\mathbb{R})$$

$$F(x)=\int_0^\infty \frac{\arctan(xy)}{y(1+y^2)}\,dy \qquad f(x,y)=\frac{\arctan(xy)}{y(1+y^2)}$$

- $F(0)=\int_0^\infty 0\,dy=0 \Rightarrow$ za $x=0$ F obstaja ✓
- $f_x(x,y)=\frac{1}{\cancel{y}(1+y^2)}\cdot\frac{1}{1+(xy)^2}\cdot\cancel{y}=\frac{1}{(1+y^2)(1+x^2y^2)}$

  f_x obstaja za $x\in\mathbb{R}$, $y\in(0,\infty)$

  f_x zvezen v obeh spremenljivkah

- $$f_x(x,y)=\frac{1}{(1+y^2)(1+x^2y^2)}\leq\frac{1}{1+y^2}$$

  $$\int_0^\infty\frac{1}{1+y^2}\,dy=\arctan y\Big|_0^\infty=\frac{\pi}{2}$$

  $\Rightarrow g(y)=\frac{1}{1+y^2}$ in $\int_0^\infty g(y)\,dy$ konvergira

  $\Rightarrow \int_0^\infty f_x(x,y)\,dy$ konvergira enakomerno

$\Rightarrow$ $F'(x)=\int_0^\infty f_x(x,y)\,dy$ če $\int_0^\infty f_x(x,y)\,dy$ obstaja

$$f_x(x,y)=\frac{1}{(1+y^2)(1+x^2y^2)}=\frac{A+By}{1+y^2}+\frac{C+Dy}{1+x^2y^2}$$

$$1=(A+By)(1+x^2y^2)+(C+Dy)\cdot(1+y^2)$$

$$1=A+By+Ax^2y^2+Bx^2y^3+C+Dy+Cy^2+Dy^3$$

y^0: $1 = A + C \longrightarrow C = 1 - A$

y^1: $0 = B + D \longrightarrow D = -B$

y^2: $0 = Ax^2 + C = Ax^2 + 1 - A = A(x^2-1) + 1 \longrightarrow \boxed{A = \frac{1}{1-x^2}} \Rightarrow C = \frac{1-x^2-1}{1-x^2} \Rightarrow \boxed{C = \frac{x^2}{1-x^2}}$

y^3: $0 = Bx^2 + D = Bx^2 - B = B \cdot (x^2-1) \longrightarrow \boxed{B=0} \Rightarrow \boxed{D=0}$

za $x \in \mathbb{R} \setminus \{-1, 1\}$!

$$f_x(x,y) = \frac{1}{1-x^2} \cdot \frac{1}{1+y^2} - \frac{x^2}{1-x^2} \cdot \frac{1}{1+x^2y^2} \quad \text{za } x \in \mathbb{R} \setminus \{-1,1\}$$

$$F'(x) = \int_0^\infty \frac{1}{1-x^2} \left(\frac{1}{1+y^2} - \frac{x^2}{1+x^2y^2} \right) dy =$$

glej nalogo zgoraj!

$$= \frac{1}{1-x^2} \cdot \left(\int_0^\infty \frac{1}{1+y^2} \, dy - x \int_0^\infty \frac{x}{1+x^2y^2} \, dy \right) =$$

$$= \frac{1}{1-x^2} \cdot \left(\arctan y \Big|_0^\infty - x \cdot \begin{cases} \arctan(xy) \Big|_0^\infty & ; x > 0 \\ 0 & ; x = 0 \\ \arctan(xy) \Big|_0^{-\infty} & ; x < 0 \end{cases} \right) =$$

$$= \frac{1}{1-x^2} \cdot \left(\frac{\pi}{2} - 0 - x \cdot \begin{cases} \frac{\pi}{2} - 0 & ; x > 0 \\ 0 & ; x = 0 \\ 0 - \frac{\pi}{2} & ; x < 0 \end{cases} \right) =$$

$$= \frac{1}{1-x^2} \cdot \left(\frac{\pi}{2} - |x| \cdot \frac{\pi}{2} \right) =$$

$$= \frac{\pi}{2} \cdot \frac{1}{1-x^2} \cdot (1 - |x|) = \frac{\pi}{2} \cdot \frac{1-|x|}{1-x^2} = \frac{\pi}{2} \cdot \frac{1-|x|}{1-|x|^2} = \frac{\pi}{2} \cdot \frac{\cancel{1-|x|}}{\cancel{(1-|x|)} \cdot (1+|x|)} =$$

$$= \underline{\underline{\frac{\pi}{2 \cdot (1+|x|)}}}$$

$\Rightarrow F'(x) = \dfrac{\pi}{2(1+|x|)}$ za $x \in \mathbb{R} \setminus \{-1,1\}$

upoštevamo zveznost integrala s parametrom:

$$\left.\begin{aligned} &f_x(x,y) = \frac{1}{(1+y^2)(1+x^2y^2)} \text{ zvezna} \\ &\int_0^\infty f_x(x,y)\,dy \text{ konvergira enakomerno} \end{aligned}\right\} \Rightarrow \int_0^\infty f_x(x,y)\,dy \text{ zvezna}$$

$\Rightarrow F'(x)$ je zvezna funkcija

$\Rightarrow F'(x) = \dfrac{\pi}{2\cdot(1+|x|)}$ za $\forall x \in \mathbb{R}$

$x \geq 0$: $\quad F'(x) = \dfrac{\pi}{2\cdot(1+x)} \Rightarrow F(x) = \dfrac{\pi}{2} \cdot \ln(1+x) + C$

$$F(0) = 0 \Rightarrow \frac{\pi}{2} \cdot \ln(1+0) + C = 0 \Rightarrow C = 0$$

$x \leq 0$: $\quad F'(x) = \dfrac{\pi}{2\cdot(1-x)} \Rightarrow F(x) = -\dfrac{\pi}{2} \cdot \ln(1-x) + D$

$$F(0) = 0 \Rightarrow -\frac{\pi}{2} \cdot \ln(1-0) + D = 0 \Rightarrow D = 0$$

$$F(x) = \begin{cases} \frac{\pi}{2} \cdot \ln(1+x) & ; x \geq 0 \\ -\frac{\pi}{2} \cdot \ln(1-x) & ; x \leq 0 \end{cases}$$

$$\underline{\underline{F(x) = \operatorname{sgn}(x) \cdot \frac{\pi}{2} \cdot \ln(1+|x|)}}$$

Naloga: Izračunajte integral $F(a)=\int_{-\infty}^{\infty}\frac{\ln(1+a^2x^2)}{1+x^2}dx$. Kot znano lahko privzamete, da je integral $F(a)$ enakomerno konvergenten in da lahko odvajate pod integralskim znakom.

$$F(a)=\int_{-\infty}^{\infty} f(a,x)\,dx \qquad f(a,x)=\frac{\ln(1+a^2x^2)}{1+x^2} \qquad f_a(a,x)=\frac{1}{1+x^2}\cdot\frac{1}{1+a^2x^2}\cdot(2ax^2)$$

Če je $a>0$ in $a\neq 1$, velja: $\int_{-\infty}^{\infty} f_a(a,x)\,dx = 2a\cdot\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}\cdot\frac{1}{1+a^2x^2}dx = *$

$$\frac{x^2}{(1+x^2)\cdot(1+a^2x^2)}=\frac{Ax+B}{1+x^2}+\frac{Cx+D}{1+a^2x^2}=\frac{Ax+B+Aa^2x^3+Ba^2x^2+Cx+D+Cx^3+Dx^2}{(1+x^2)(1+a^2x^2)}$$

x^0: $0=B+D \Rightarrow D=-B$

x^1: $0=A+C \Rightarrow C=-A$

x^2: $1=Ba^2+D \longrightarrow 1=Ba^2-B \Rightarrow B\cdot(a^2-1)=1 \Rightarrow \boxed{B=\frac{1}{a^2-1}}$

x^3: $0=Aa^2+C \longrightarrow 0=Aa^2-A \Rightarrow A\cdot(a^2-1)=0 \Rightarrow \boxed{A=0}$

$\boxed{D=\frac{-1}{a^2-1}}$ $\boxed{C=0}$ tukaj potrebujemo $a\neq -1,1$

$$* = 2a\cdot\int_{-\infty}^{\infty}\frac{1}{a^2-1}\cdot\frac{1}{1+x^2}dx - 2a\cdot\int_{-\infty}^{\infty}\frac{1}{a^2-1}\cdot\frac{1}{1+a^2x^2}dx =$$

$$=\frac{2a}{a^2-1}\cdot\arctan x\Big|_{-\infty}^{\infty}-\frac{2a}{a^2-1}\cdot\int_{-\infty}^{\infty}\frac{1}{1+(ax)^2}dx=$$

tukaj upoštevamo $a>0$!

$$=\frac{2a}{a^2-1}\cdot\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)-\frac{2\cancel{a}}{a^2-1}\cdot\arctan(ax)\cdot\frac{1}{\cancel{a}}\Big|_{x=-\infty}^{\infty}=$$

$$=\frac{2a\pi}{a^2-1}-\frac{2}{a^2-1}\cdot\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\frac{2\pi}{a^2-1}\cdot(a-1)=\frac{2\pi\cdot\cancel{(a-1)}}{\cancel{(a-1)}\cdot(a+1)}=\frac{2\pi}{a+1} \quad \text{za } a\neq 1$$

$\Rightarrow F'(a)=\int_{-\infty}^{\infty} f_a(a,x)\,dx$ po predpostavki

$\Rightarrow F(a)=\int F'(a)\,da=\int\frac{2\pi}{a+1}da=2\pi\cdot\ln(a+1)+C$ za $a\neq 1$

f je zvezna ker $1+x^2\neq 0$ in $1+a^2x^2>0$, F enakomerno konv. po predpostavki $\Rightarrow F$ zvezna

F zvezna v $a=1 \Rightarrow F(a)=2\cdot\pi\cdot\ln(a+1)+C$ za $\forall a\geq 0$

F zvezna v $a=0 \Rightarrow F(0)=\int_{-\infty}^{\infty}\frac{\ln(1+0^2x^2)}{1+x^2}dx=0$

$F(0)=2\cdot\pi\cdot\ln(0+1)+C=C$

$\Big\}\; C=0$

$\Rightarrow F(a)=2\cdot\pi\cdot\ln(a+1)$ za $a\geq 0$

Ker po definiciji velja $F(-a)=F(a)$ (F je soda funkcija), velja $\underline{F(a)=2\pi\cdot\ln(|a|+1)}$ za $a\in\mathbb{R}$.

INTEGRIRANJE INTEGRALA S PARAMETROM

$$F(x) = \int_a^\infty f(x,t)\,dt$$

f zvezna na $[c,d] \times [a,\infty)$ in $\int_a^\infty f(x,t)\,dt$ enakomerno konvergira na $[c,d]$

$\underset{x}{\uparrow}$ $\quad$ $\underset{t}{\uparrow}$

$$\Rightarrow \int_c^d F(x)\,dx = \int_c^d \left(\int_a^\infty f(x,t)\,dt \right) dx = \int_a^\infty \left(\int_c^d f(x,t)\,dx \right) dt$$

<u>Naloga:</u> Izračunaj $\int_0^\infty \frac{e^{-at} - e^{-2at}}{t}\,dt$ za $a > 0$.

$$\frac{e^{-at} - e^{-2at}}{t} = \frac{e^{-xt}}{t}\Big|_{x=2a}^{a} = \int_{2a}^{a} \frac{1}{\cancel{t}} \cdot e^{-xt} \cdot (-\cancel{t})\,dx = \int_a^{2a} e^{-xt}\,dx$$

$$\int_0^\infty \frac{e^{-at} - e^{-2at}}{t}\,dt = \int_0^\infty \left(\int_a^{2a} e^{-xt}\,dx \right) dt$$

<u>želimo, da je to enako $\int_a^{2a} \left(\int_0^\infty e^{-xt}\,dt \right) dx.$</u>

Opazujmo integral $F(x) = \int_0^\infty e^{-xt}\,dt$

$$f(x,t) = e^{-xt}$$

$$|f(x,t)| = |e^{-xt}| = e^{-xt} \leq \overset{g(t)}{e^{-at}} \quad \text{za } t \in [0,\infty) \text{ in } x \in [a,2a] \quad (\text{ker je } a>0 \Rightarrow a < 2a)$$

$$\int_0^\infty g(t)\,dt = \int_0^\infty e^{-at}\,dt = \frac{e^{-at}}{-a}\Big|_{t=0}^{\infty} = -\frac{1}{a}(0-1) = \frac{1}{a} < \infty$$

$\left.\begin{array}{l} \Rightarrow \int_0^\infty e^{-xt}\,dt \text{ konvergira enakomerno na } [a,2a] \\ f(x,t) = e^{-xt} \text{ je zvezna na } [a,2a] \times [0,\infty) \end{array}\right\} \int_a^{2a} \underbrace{\left(\int_0^\infty e^{-xt}\,dt \right)}_{F(x)} dx \overset{*}{=} \underbrace{\int_0^\infty \left(\int_a^{2a} e^{-xt}\,dx \right) dt}_{\text{iskan integral}}$

$$\Rightarrow \int_0^\infty \frac{e^{-at} - e^{-2at}}{t}\,dt = \int_0^\infty \left(\int_a^{2a} e^{-xt}\,dx \right) dt \overset{*}{=} \int_a^{2a} \left(\int_0^\infty e^{-xt}\,dt \right) dx = \int_a^{2a} \frac{e^{-xt}}{-x}\Big|_{t=0}^{\infty} dx = \int_a^{2a} \left(0 - \frac{1}{-x}\right) dx =$$

$$= \int_a^{2a} \frac{1}{x}\,dx = \ln|x|\Big|_{x=a}^{2a} = \ln(2a) - \ln(a) = \ln\frac{2\cancel{a}}{\cancel{a}} = \underline{\underline{\ln 2}}$$

FUNKCIJI GAMA IN BETA

Funkcija GAMA: $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\,dt \qquad x>0$

$$\Gamma(n+1) = \int_0^\infty t^n \cdot e^{-t}\,dt \underset{\substack{\uparrow \\ \text{per partes}}}{=} \ldots = n\cdot\Gamma(n) = n!$$

$$\Gamma(x+1) = x\cdot\Gamma(x)$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Naloga: Izračunaj $\int_0^\infty x^7 \cdot e^{-\frac{x^2}{2}}\,dx$

$$\int_0^\infty x^7 \cdot e^{-\frac{x^2}{2}}\,dx \underset{\uparrow}{=} \int_0^\infty (x^2)^3 \cdot \cancel{x} \cdot e^{-t} \cdot \frac{dt}{\cancel{x}} = \int_0^\infty (2t)^3 \cdot e^{-t}\,dt = 8\cdot\int_0^\infty t^3 \cdot e^{-t}\,dt =$$

$$t = \frac{x^2}{2} \rightarrow x^2 = 2t$$

$$dt = \cancel{\tfrac{1}{2}}\cdot\cancel{2}x\,dx \rightarrow dx = \frac{dt}{x}$$

$$= 8\cdot\Gamma(4) = 8\cdot 3! = \underline{\underline{48}}$$

Naloga: Izračunaj $\int_0^1 (\ln x)^4 \cdot \sqrt{-\ln x}\,dx.$

$$\int_0^1 (\ln x)^4 \cdot \sqrt{-\ln x}\,dx = \int_\infty^0 (-t)^4 \cdot \sqrt{t} \cdot (-x)\,dt = \int_\infty^0 t^4 \cdot t^{\frac{1}{2}} \cdot (-e^{-t})\,dt =$$

$$t = -\ln x \qquad x\to 0:\ \ln x \to -\infty,\ -\ln x \to \infty$$

$$dt = -\frac{1}{x}\,dx \rightarrow dx = -x\,dt$$

$$\ln x = -t \Rightarrow x = e^{-t}$$

$$= -\int_\infty^0 t^{4+\frac{1}{2}} e^{-t}\,dt = \int_0^\infty t^{\frac{9}{2}} e^{-t}\,dt =$$

$$= \Gamma\left(\tfrac{9}{2}+1\right) = \Gamma\left(\tfrac{11}{2}\right) = \frac{9}{2}\cdot\Gamma\left(\tfrac{9}{2}\right) = \frac{9\cdot 7}{2^2}\cdot\Gamma\left(\tfrac{7}{2}\right) =$$

$$= \frac{9\cdot7\cdot5}{2^3}\cdot\Gamma\left(\tfrac{5}{2}\right) = \frac{9\cdot7\cdot5\cdot3}{2^4}\cdot\Gamma\left(\tfrac{3}{2}\right) = \frac{9\cdot7\cdot5\cdot3\cdot1}{2^5}\cdot\Gamma\left(\tfrac{1}{2}\right) =$$

$$= \underline{\underline{\frac{945}{32}\cdot\sqrt{\pi}}}$$

Naloga: Izračunaj $\int_1^\infty \frac{1}{x^2}\sqrt{\ln x}\,dx$.

$$\int_1^\infty \frac{1}{x^2}\sqrt{\ln x}\,dx \overset{\uparrow}{=} \int_0^\infty \frac{1}{e^{\cancel{2t}}}\cdot\sqrt{t}\,\cancel{e^t}\,dt = \int_0^\infty t^{\frac{1}{2}} e^{-t}\,dt = \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\cdot\Gamma\left(\tfrac{1}{2}\right) = \underline{\underline{\frac{\sqrt{\pi}}{2}}}$$

$t = \ln x \rightarrow x = e^t$

$dt = \frac{1}{x}dx \rightarrow dx = x\,dt = e^t dt$

Naloga: Izračunaj $\int_0^1 \sqrt{\left(\ln\frac{1}{t}\right)^3}\,dt$.

$$\int_0^1 \sqrt{\left(\ln\tfrac{1}{t}\right)^3}\,dt \overset{\uparrow}{=} \int_\infty^0 \sqrt{x^3}\,(-e^{-x})\,dx = -\int_\infty^0 x^{\frac{3}{2}} e^{-x}\,dx = \int_0^\infty x^{\frac{3}{2}} e^{-x}\,dx =$$

$x = \ln\frac{1}{t} \rightarrow e^x = \frac{1}{t} \rightarrow t = e^{-x}$

$dx = \frac{1}{\frac{1}{t}}\cdot\left(-\frac{1}{t^2}\right)dt = -\frac{\cancel{t}}{t^{\cancel{2}}}\,dt = -\frac{1}{t}\,dt \rightarrow dt = -t\,dx$

$dt = -e^{-x}dx$

$t \rightarrow 0 \Rightarrow \frac{1}{t} \rightarrow \infty \Rightarrow \ln\frac{1}{t} \rightarrow \infty$

$t \rightarrow 1 \Rightarrow \frac{1}{t} \rightarrow 1 \Rightarrow \ln\frac{1}{t} \rightarrow 0$

$$= \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2}\cdot\Gamma\left(\tfrac{3}{2}\right) = \tfrac{3}{4}\cdot\Gamma\left(\tfrac{1}{2}\right) = \underline{\underline{\tfrac{3}{4}\sqrt{\pi}}}$$

Funkcija BETA:

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

$$B(x,y)=\frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$$

$$B(x,y)=\int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,du$$

$$B(x,y)=2\cdot\int_0^{\frac{\pi}{2}} \cos^{2x-1}\varphi\cdot\sin^{2y-1}\varphi\;d\varphi$$

Naloga: Izračunaj $\int_0^1\sqrt{x-x^2}\,dx$.

$$\int_0^1\sqrt{x-x^2}\,dx=\int_0^1\sqrt{x}\cdot\sqrt{1-x}\,dx=\int_0^1 x^{\frac12}(1-x)^{\frac12}\,dx=$$

$$=\int_0^1 x^{\frac32-\frac12}\cdot(1-x)^{\frac32-\frac12}\,dx=$$

$$=B\left(\tfrac32,\tfrac32\right)=\frac{\Gamma(\frac32)\cdot\Gamma(\frac32)}{\Gamma(\frac32+\frac32)}=\frac{\frac12\Gamma(\frac12)\cdot\frac12\Gamma(\frac12)}{\Gamma(3)}=$$

$$=\frac14\cdot\frac{(\sqrt{\pi})^2}{2!}=\frac{\pi}{8}$$

***Naloga:** Izrazi $\int_0^1\frac{1}{\sqrt{1-x^3}}\,dx$ z beta funkcijo.

$$\int_0^1\frac{1}{\sqrt{1-x^3}}\,dx\underset{\substack{t=x^3 \\ dt=3x^2dx}}{=}\int_0^1(1-t)^{-\frac12}\frac{dt}{3x^2}=\frac13\int_0^1(1-t)^{-\frac12}\cdot\left(t^{\frac13}\right)^{-2}dt=$$

$$\to x=t^{\frac13}$$

$$=\frac13\int_0^1 t^{-\frac23}(1-t)^{-\frac12}\,dt=\frac13\int_0^1 t^{\frac13-1}(1-t)^{\frac12-1}\,dt=$$

$$=\frac13\cdot B\left(\tfrac13,\tfrac12\right)$$

Naloga: Izračunaj $\int_0^4 x^3 (16-x^2)^{\frac{1}{2}} dx$.

$$\int_0^4 x^3 (16-x^2)^{\frac{1}{2}} dx = \int_0^4 x^3 \cdot \left(16\cdot\left(1-\left(\tfrac{x}{4}\right)^2\right)\right)^{\frac{1}{2}} dx = \int_0^1 \left(4t^{\frac{1}{2}}\right)^3 \left(16\cdot(1-t)\right)^{\frac{1}{2}} \cdot 2 \cdot \frac{dt}{t^{\frac{1}{2}}} =$$

$$t = \left(\tfrac{x}{4}\right)^2 \rightarrow x = 4\sqrt{t}$$
$$dx = 4\cdot\tfrac{1}{2}\cdot t^{-\frac{1}{2}}\, dt$$

$$= \int_0^1 4^3 t^{\frac{3}{2}} \cdot 4 \cdot (1-t)^{\frac{1}{2}} \cdot 2 \cdot t^{-\frac{1}{2}} dt = 2^9 \int_0^1 t^{\frac{3}{2}-\frac{1}{2}} (1-t)^{\frac{1}{2}} dt =$$

$$= 2^9 \int_0^1 t^1 (1-t)^{\frac{1}{2}} dt = 2^9 B\left(2, \tfrac{3}{2}\right) = 2^9 \frac{\Gamma(2)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(2+\frac{3}{2}\right)} =$$

$$= 2^9 \cdot \frac{1\cdot\cancel{\frac{1}{2}}\cdot\cancel{\sqrt{\pi}}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\cancel{\frac{1}{2}}\cdot\cancel{\sqrt{\pi}}} = \underline{\underline{\frac{2^{11}}{15}}}$$

Naloga: Izračunaj $\int_0^{\frac{\pi}{2}} \sin^3 x \cdot \cos^5 x \, dx$.

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cdot \cos^5 x \, dx = \frac{1}{2}\cdot\left(2\cdot\int_0^{\frac{\pi}{2}} \cos^{2\cdot 3-1} x \cdot \sin^{2\cdot 2-1} x \, dx\right) = \frac{1}{2}\cdot B(3,2) =$$

$$\boxed{B(x,y) = 2\cdot\int_0^{\frac{\pi}{2}} \cos^{2x-1}\varphi \cdot \sin^{2y-1}\varphi \, d\varphi}$$

$$= \frac{1}{2}\cdot\frac{\Gamma(3)\cdot\Gamma(2)}{\Gamma(5)} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2!}\cdot 1!}{4!} = \underline{\underline{\frac{1}{24}}}$$

Naloga: Izračunaj $\int_0^\infty \frac{x^5}{1+x^{12}} dx$.

$$\int_0^\infty \frac{x^5}{1+x^{12}} dx = \int_0^\infty \frac{u^{\frac{5}{12}}}{1+u} \cdot \frac{du}{12\, u^{\frac{11}{12}}} = \frac{1}{12}\int_0^\infty \frac{u^{-\frac{1}{2}}}{(1+u)^1} du =$$

$$u = x^{12} \rightarrow x = u^{\frac{1}{12}}$$
$$du = 12x^{11} dx \rightarrow dx = \frac{du}{12x^{11}}$$

$$\boxed{B(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} du}$$

$$= \frac{1}{12} B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{1}{12}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{1}{12}\cdot\frac{\sqrt{\pi}\cdot\sqrt{\pi}}{\Gamma(1)} =$$

$$-\tfrac{1}{2} = x-1 \qquad x = \tfrac{1}{2}$$
$$1 = x+y \qquad y = \tfrac{1}{2}$$

$$= \frac{1}{12}\cdot\frac{\pi}{0!} = \underline{\underline{\frac{\pi}{12}}}$$