SKICIRAJ GRAF f!

$$f(x) = \frac{e^x}{x}$$

$$f(x) = \ln(x^2-1) + \frac{1}{x^2-1}$$

$$f(x) = 1 - x - \sqrt{\frac{x^3}{x+3}}$$

$$f(x) = (2x^2-17)\sqrt[3]{x^2-1}$$

$$f(x) = \frac{e^x}{x}$$

Def. območje: $\mathbb{R}\setminus\{0\}$. Funkcija ni niti soda niti liha.

Ničel funkcija nima.

$$f'(x) = \left[\frac{e^x}{x}\right]' = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$

f ima stac. točko v $x=1$. $f(1)=e$.

f je padajoča za $x<1$ in naraščajoča za $x>1$.

$$f''(x) = \frac{[e^x(x-1)]'x^2 - e^x(x-1)2x}{x^4} =$$

$$= \frac{(e^x(x-1)+e^x)x - 2e^x(x-1)}{x^3} =$$

$$= \frac{e^x[(x-1)x + x - 2x + 2]}{x^3} =$$

$$= \frac{e^x(x^2-2x+2)}{x^3} = \frac{e^x(x^2-2x+2)}{x^3}$$ brez ničel

Funkcija nima prevojev, za $x<0$ je konkavna, za $x>0$ pa konveksna.

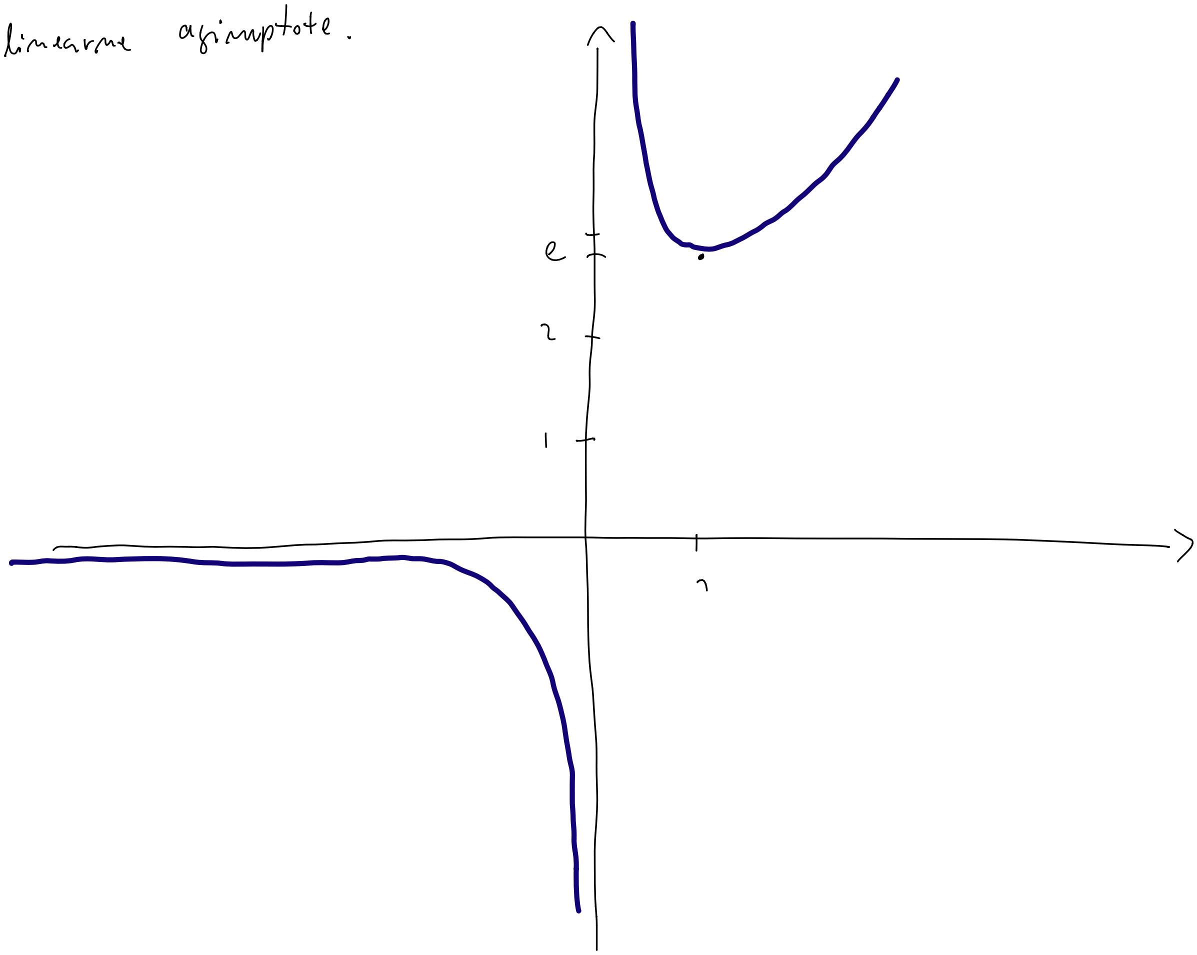
Obnašanje na robovih def. območja

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{e^x}{x} = 0$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x}{x} \ \ldots \ \lim_{x\uparrow 0} \frac{e^x}{x} = -\infty, \ \lim_{x\downarrow 0} \frac{e^x}{x} = \infty$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x}{x} \overset{L'H}{=} \lim_{x\to\infty} e^x = \infty$$

Ker je tudi $\lim_{x\to\infty} f'(x) = \infty$, vidimo, da funkcija nima linearne asimptote.



$$f(x) = \ln(x^2-1) + \frac{1}{x^2-1}$$

Def. območje:

$\{x \mid x^2-1 > 0\} = \{x \mid x^2 > 1\} = (-\infty,-1) \cup (1,\infty)$

Opazimo, da je f soda. Osredotočimo se zgolj na pozitivne x, torej na $x \in (1,\infty)$.

Ničle? Nima

$$f'(x) = \frac{2x}{x^2-1} - \frac{2x}{(x^2-1)^2} = 2x\,\frac{x^2-1-1}{(x^2-1)^2} =$$

$$= 2x\,\frac{x^2-2}{(x^2-1)^2} \qquad f(\sqrt{2}) = 1$$

Vidimo: f ima na $(1,\infty)$ stac. točko v $x=\sqrt{2}$, na $(1,\sqrt{2})$ pada, na $(\sqrt{2},\infty)$ pa narašča.

Ker je $f(\sqrt{2}) = 1 > 0$ in ker x na $(1,\infty)$ nima točk, v katerih ni odvedljiva, sklepamo, da f nima ničel.

$$f''(x) = \left[2x\,\frac{x^2-2}{(x^2-1)^2}\right]' =$$

$$= 2\,\frac{x^2-2}{(x^2-1)^2} + 2x\,\frac{2x(x^2-1)^{\cancel{2}} - (x^2-2)2\cancel{(x^2-1)}2x}{(x^2-1)^{\cancel{4}\,3}}$$

$$= 2\,\frac{(x^2-2)(x^2-1) + 2x^2(x^2-1) - 4x^2(x^2-2)}{(x^2-1)^3}$$

$$= 2\,\frac{x^4-3x^2+2+2x^4-2x^2-4x^4+8x^2}{(x^2-1)^3}$$

$$= 2 \frac{-x^4 + 3x^2 + 2}{(x^2-1)^3} =$$

$$= \frac{-2}{(x^2-1)^3} \left(x^4 - 3x^2 - 2 \right)$$

$t^2 - 3t - 2$ ima ničle:

$$t_{1,2} = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

Nas zanima le đigta pozitivna: $\frac{\sqrt{17}+3}{2}$

$$t = x^2$$

$$x_{1,2} = \pm \sqrt{\frac{\sqrt{17}+3}{2}}$$

za $x > 1$

$$f(x_1) = \ln\left(\frac{\sqrt{17}+3}{2} - 1\right) + \frac{1}{\frac{\sqrt{17}+3}{2} - 1} \approx 1,4$$

f ima prevoj v $\sqrt{\frac{\sqrt{17}+3}{2}} \approx 1.9$

f je konkavna za $x > \sqrt{\frac{\sqrt{17}+3}{2}}$

f je konvekgna za $x < \sqrt{\frac{\sqrt{17}+3}{2}}$

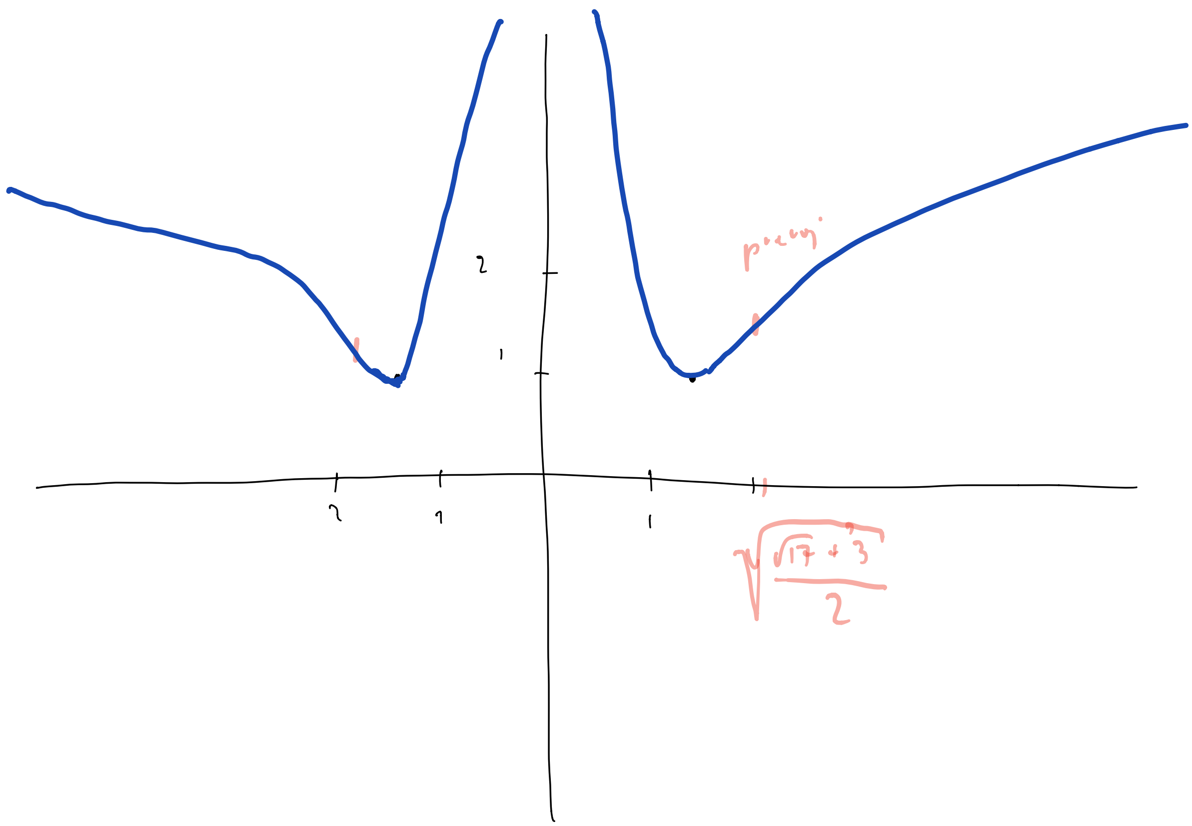
Obnašanje na robovih def. območja:

$$\lim_{x \downarrow 1} f(x) = \lim_{x \downarrow 1} \underbrace{\ln(x^2-1)}_{\to -\infty} + \underbrace{\frac{1}{x^2-1}}_{\to \infty} =$$

$$= \lim_{x \downarrow 1} \underbrace{\frac{1}{x^2-1}}_{\downarrow\ \infty} \Big[\underbrace{\ln(x^2-1)\cdot(x^2-1)}_{\downarrow\ 0} + 1 \Big] = \infty$$

$$\lim_{t \downarrow 0} t \cdot \ln t = \lim_{t \downarrow 0} \frac{\ln t}{\frac{1}{t}} \overset{L'H}{=} \lim_{t \downarrow 0} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = 0$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \underbrace{\ln(x^2-1)}_{\downarrow\ \infty} + \underbrace{\frac{1}{x^2-1}}_{\downarrow\ 0} = \infty$$



$$f(x) = 1 - x - \sqrt{\frac{x^3}{x+3}}$$

Def. območje:

$$\left\{ x \mid x \neq -3 \text{ in } \frac{x^3}{x+3} > 0 \right\}$$

$$\frac{x^3}{x+3} > 0 \iff x^3(x+3) > 0 \iff x(x+3) > 0,$$

tj. za $x \in (-\infty, -3) \cup [0, \infty) = D_f$

↳ ničla v imenovalcu

f ni niti soda niti liha. $f(0) = 1$

Ničle

$$1 - x - \sqrt{\frac{x^3}{x+3}} = 0$$

$$1 - x = \sqrt{\frac{x^3}{x+3}} \quad \Big/ ()^2 \quad (*)$$

$$(1-x)^2 = \frac{x^3}{x+3}$$

$$(1-x)^2(x+3) = x^3$$

$$(1 - 2x + x^2)(x+3) = x^3$$

$$x^3 + x^2 - 5x + 3 = x^3$$

$$x^2 - 5x + 3 = 0$$

$$x_{1,2} = \frac{5 \pm \sqrt{25-12}}{2}$$

$$x_{1,2} = \frac{5 \pm \sqrt{13}}{2}$$ ... kandidata za ničle

Preizkus: pri kvadriranju (*) smo lahko dobili še eno rešitev. Katera je prava.

$$f'(x) = -1 - \frac{1}{2}\frac{1}{\sqrt{\frac{x^3}{x+3}}}\frac{3x^2(x+3)-x^3}{(x+3)^2}$$

$$= -1 - \frac{1}{2}\frac{(x+3)^{\frac{1}{2}}}{x^{3/2}}\cdot\frac{x^2(3x+9-x)}{(x+3)^2} =$$

$2-\frac{3}{2}=\frac{1}{2}$

$2-\frac{1}{2}=\frac{3}{2}$

$$= -1 - \frac{1}{2}(2x+9)\frac{x^{2-\frac{3}{2}}}{(x+3)^{2-\frac{1}{2}}} =$$

$$= -1 - \frac{1}{2}(2x+9)\sqrt{\frac{x}{(x+3)^3}}$$

f na def. območju nima stac. točk.

Na D_f vidimo, da je $f'<0$, torej je f vedno padajoča.

Ničle?

$f(0)=1$, $f(3) = 1-3-\sqrt{\frac{3^3}{6}} < 0$, $\Rightarrow$

$\hookrightarrow$ med x_1 in x_2

f je padajoča za $x>0$

$\Rightarrow$ f ima natanko eno ničlo med 0 in 3, to pa je $\frac{5-\sqrt{13}}{2}$

$$f''(x) = -\frac{1}{2}\cdot 2\sqrt{\frac{x}{(x+3)^3}} - \frac{1}{2}(2x+9)\frac{1}{2}\sqrt{\frac{(x+3)^3}{x}}\,\frac{3-2x}{(x+3)^4}$$

$$\left(\frac{x}{(x+3)^3}\right)' = \frac{(x+3)^3 - x(x+3)^2\cdot 3}{(x+3)^6} =$$

$$= \frac{x+3-x\cdot 3}{(x+3)^4} =$$

$$= \frac{3-2x}{(x+3)^4}$$

$$= \underbrace{-\sqrt{\frac{x}{(x+3)^3}}}_{\text{vedno neg.}} - \frac{1}{4}(2x+9)(3-2x)\frac{1}{\sqrt{x}\,(x+3)^{4-\frac{3}{2}}}$$

$$= \cdots = -\frac{27x^4}{4(x+3)^4\left(\frac{x^3}{x+3}\right)^{3/2}}$$

Vidimo, da je f vedno konkavna na D_f

Obnašanje na robovih def. območja:

$$\lim_{x \nearrow -3} \underbrace{1 - x}_{4} - \underbrace{\sqrt{\frac{x^3}{x+3}}}_{\infty} = -\infty$$

$$\lim_{x \to \infty} 1 - \underbrace{x}_{\infty} - \underbrace{\sqrt{\frac{x^3}{x+3}}}_{\infty} = -\infty$$

$$\lim_{x \to -\infty} 1 - x - \sqrt{\frac{x^3}{x+3}} \overset{t := -x}{=}$$

$$= \lim_{t \to \infty} 1 + t - \sqrt{\frac{(-t)^3}{-t+3}} = \lim_{t \to \infty} 1 + t - \sqrt{\frac{t^3}{t-3}} =$$

$$= \lim_{t \to \infty} \frac{(1+t)\sqrt{t-3} - t\sqrt{t}}{\sqrt{t-3}} =$$

$$t(\sqrt{t-3} - \sqrt{t})$$

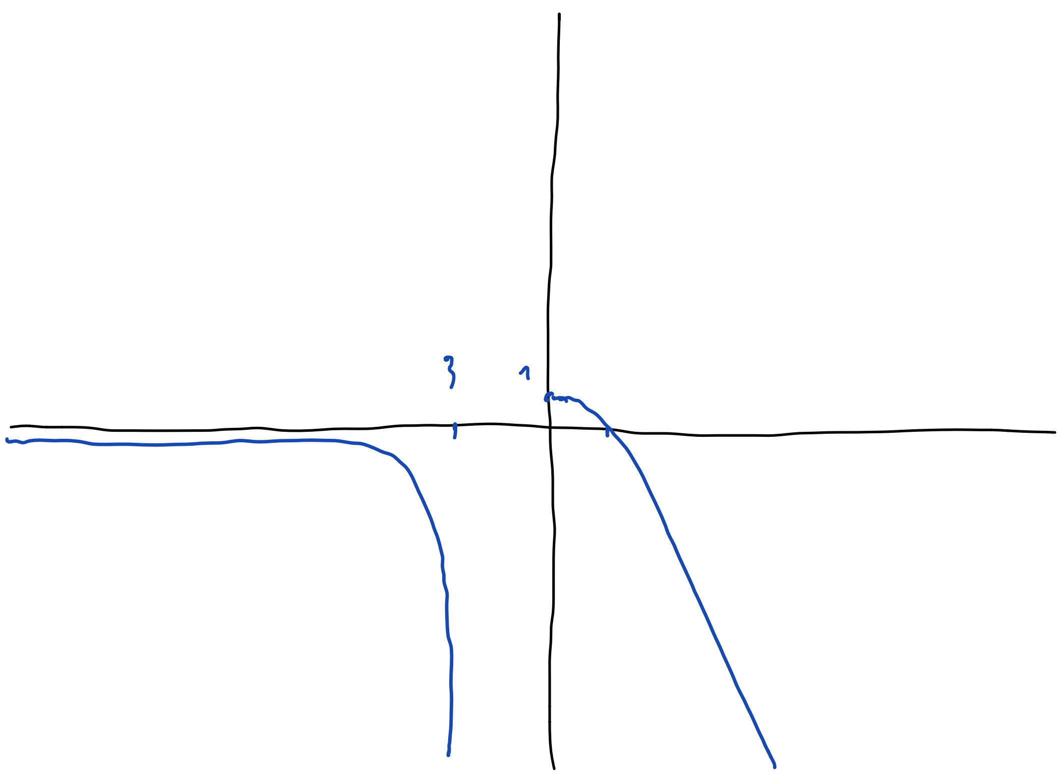
$$= \lim_{t \to \infty} 1 + \frac{[illegible]}{\sqrt{t-3}} =$$

$$= \lim_{t \to \infty} 1 + \frac{-3t}{\sqrt{t-3}\,(\sqrt{t-3} + \sqrt{t})} =$$

$$= 1 + \lim_{t \to \infty} \frac{-3t}{t - 3 + \sqrt{t(t-3)}} = -\frac{1}{2}$$

$$\frac{3t}{t-3+\sqrt{(t-3)^2}} \leq \frac{3t}{t-3+\sqrt{t(t-3)}} \leq \frac{3t}{t-3+\sqrt{t^2}}$$

$$\frac{3t}{t-3+\sqrt{(t-3)^2}} \to \frac{3}{2} \qquad \frac{3t}{t-3+\sqrt{t^2}} \to \frac{3}{2}$$



$$f(x) = (2x^2 - 17)\sqrt[3]{x^2 - 1}$$

Def. območje : $\mathbb{R}$

f je soda $\Rightarrow$ osredotočimo se na $[0, \infty)$

$f(0) = 17$, Ničle: $x_1 = 1$, $x_2 = \sqrt{\frac{17}{2}} \approx \sqrt{8}$

$$f'(x) = 4x\sqrt[3]{x^2 - 1} + (2x^2 - 17)\frac{1}{3}\frac{2x}{(x^2-1)^{2/3}} =$$

$$= 2x\left(2\sqrt[3]{x^2-1} + \frac{1}{3}\frac{2x^2-17}{(x^2-1)^{2/3}}\right) =$$

$$= 2x\,\frac{6(x^2-1) + 2x^2 - 17}{3(x^2-1)^{2/3}} =$$

$$= 2x\,\frac{6x^2 - 6 + 2x^2 - 17}{3(x^2-1)^{2/3}} =$$

$$= 2x\,\frac{8x^2 - 23}{3(x^2-1)^{2/3}}$$

imenovalec je vedno ≥ 0

f ni odvedljiva v $x = 1$, $f(1) = 0$

f ima stac. točko v $x = 0$ in $x = \sqrt{\frac{23}{8}} \approx \sqrt{3}$

f zamenja predznak v stac. točkah, ne pa v točki odvedljivosti.

$f'(x) > 0$ za $x > \sqrt{\frac{23}{8}}$, $f'(x) < 0$ za $x \in \left(0, \sqrt{\frac{23}{8}}\right)$

$$f\left(\sqrt{\tfrac{23}{8}}\right) = \left(2\cdot\frac{23}{8} - 17\right)\sqrt[3]{-\frac{1}{8}} = -\frac{1}{2}\,\frac{23 - 4\cdot 17}{4} = \underline{\underline{\frac{45}{8}}}$$

$$f''(x) = \left[2x\,\frac{8x^2-23}{3(x^2-1)^{2/3}}\right]' =$$

$$= \left[\frac{2}{3}\,\frac{8x^3-23x}{(x^2-1)^{2/3}}\right]' =$$

$$= \frac{2}{3}\,\frac{(24x^2-23)(x^2-1)^{2/3} - (8x^3-23x)\frac{2}{3}(x^2-1)^{-1/3}2x}{(x^2-1)^{4/3}}$$

$$= \frac{2}{3}\,\frac{3(24x^2-23)(x^2-1) - 4x(8x^3-23x)}{3(x^2-1)^{5/3}} =$$

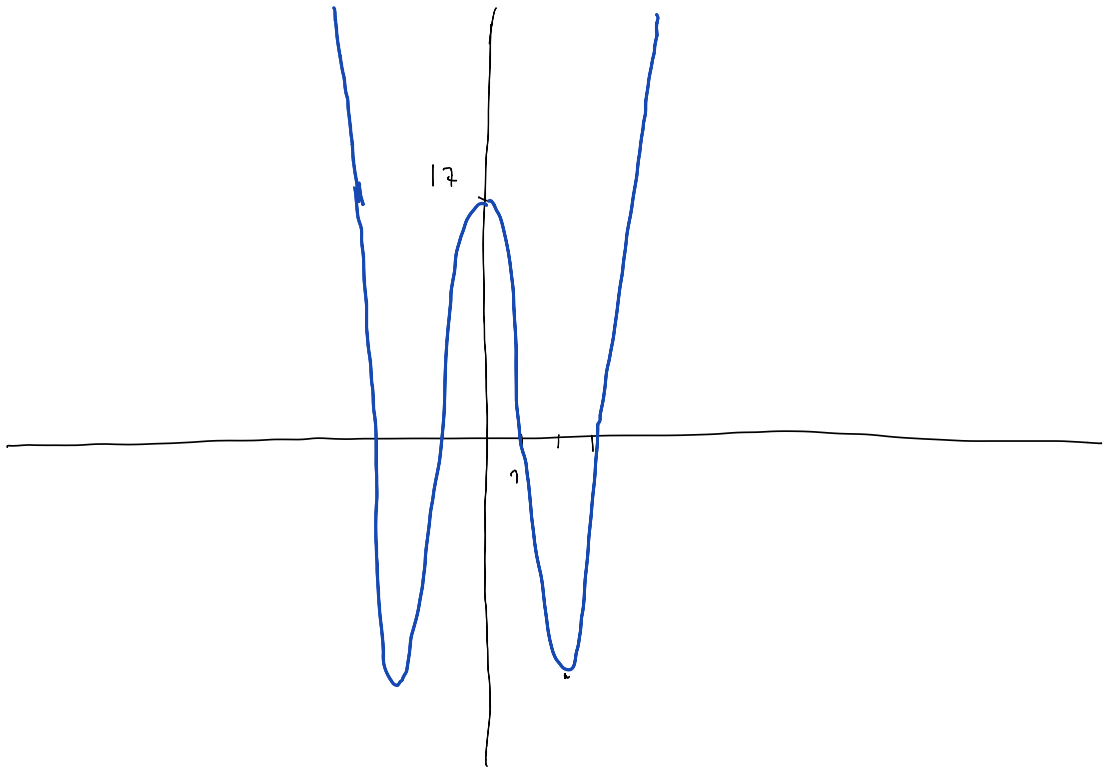
$$= \frac{2}{9}\,\frac{40x^4-49x^2+69}{(x^2-1)^{5/3}} = \underline{\underline{\frac{80x^4-98x^2+138}{9(x^2-1)^{5/3}}}}$$

$98^2 - 4\cdot 80\cdot 138 < 0 \Rightarrow f''$ nima ničel, a zamenja predznak v točki $x=1$.

$f''(x) > 0$ za $x > 1$, $f''(x) < 0$ za $x < 1$

$$\lim_{x\to\infty} f(x) = \infty$$

$\lim_{x\to\infty} f'(x) = \infty \Rightarrow f$ nima poševne asimptote


17