Kvaternioni so matrike oblike

$$\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$$

kjer $\alpha, \beta \in \mathbb{C}$

Pokažimo, da je to podkolobar v 2×2 matrikah nad $\mathbb{C}$

Zaprtost za odštevanje

$$\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} - \begin{bmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma} \end{bmatrix} = \begin{bmatrix} \alpha - \gamma & \beta - \delta \\ -\bar{\beta} + \bar{\delta} & \bar{\alpha} - \bar{\gamma} \end{bmatrix} =$$

$$= \begin{bmatrix} \alpha - \gamma & \beta - \delta \\ -\overline{(\beta - \gamma)} & \overline{\alpha - \gamma} \end{bmatrix}$$

Zaprtost za množenje

$$\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \begin{bmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma} \end{bmatrix} = \begin{bmatrix} \alpha\gamma - \beta\bar{\delta} & \alpha\delta + \beta\bar{\gamma} \\ -\bar{\beta}\gamma - \bar{\alpha}\bar{\delta} & -\bar{\beta}\delta + \bar{\alpha}\bar{\gamma} \end{bmatrix}$$

Velja $\overline{\alpha\gamma - \beta\bar{\delta}} = \bar{\alpha}\bar{\gamma} - \bar{\beta}\delta =$

in $-\overline{(\alpha\delta + \beta\bar{\gamma})} = -(\bar{\alpha}\bar{\delta} + \bar{\beta}\gamma) = -\bar{\alpha}\bar{\delta} - \bar{\beta}\gamma =$

· Dokazali smo, da so matrike oblike

$$(*) \quad \begin{bmatrix} \alpha & z \\ -\bar{z} & \bar{\alpha} \end{bmatrix}$$

kolobar za običajni + in ·.

Pokažimo še, da je ta kolobar obseg

Vsaka neničelna matrika oblike (*) ima inverz, ki je spet oblike (*).

Spomnimo se, da je $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

V našem primeru dobimo

$$\begin{bmatrix} \alpha & z \\ -\bar{z} & \bar{\alpha} \end{bmatrix}^{-1} = \frac{1}{\alpha\bar{\alpha} + z\bar{z}} \begin{bmatrix} \bar{\alpha} & -z \\ \bar{z} & \alpha \end{bmatrix}$$

$$= \frac{1}{\alpha\bar{\alpha} + z\bar{z}} \begin{bmatrix} \bar{\alpha} & -z \\ \bar{z} & \alpha \end{bmatrix}$$

Opazimo, da je $\underbrace{\alpha\bar{\alpha} + z\bar{z}}_{=\, a^2+b^2+c^2+d^2}$ je realno število

$\alpha = a+bi$
$z = c+di$ $\Rightarrow$

Ker je $\begin{bmatrix} \alpha & z \\ -\bar{z} & \bar{\alpha} \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, je $a^2+b^2+c^2+d^2 \neq 0$, ker je vsaj eden od a, b, c, d različen od 0

Označimo $\gamma = \dfrac{\bar\alpha}{\alpha\bar\alpha + \beta\bar\beta}$, $\delta = \dfrac{-\beta}{\alpha\bar\alpha + \beta\bar\beta}$

Opazimo, da je $\begin{bmatrix}\alpha & \beta \\ -\bar\beta & \bar\alpha\end{bmatrix}^{-1} = \begin{bmatrix}\gamma & \delta \\ -\bar\delta & \bar\gamma\end{bmatrix}$

Torej je tudi inverz kvaterniona spet kvaternion.

$$0 \cdot x = 0$$

$$0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x \quad -0 \cdot x$$

$$= 0 + 0 \cdot x$$

Odštejemo $0 \cdot x$ na obeh straneh in dobimo

$$0 = 0 \cdot x$$

$$\circ = + \implies a \circ b^{-1} = a - b$$

$$\circ = \cdot \implies a \circ b^{-1} = a/b$$