Problem brahistohrone (nadaljevanje)

Obravnavamo funkcijo

$$g(\theta) = 1 - \cos\theta + \frac{B}{b}(\theta - \sin\theta) = 0$$

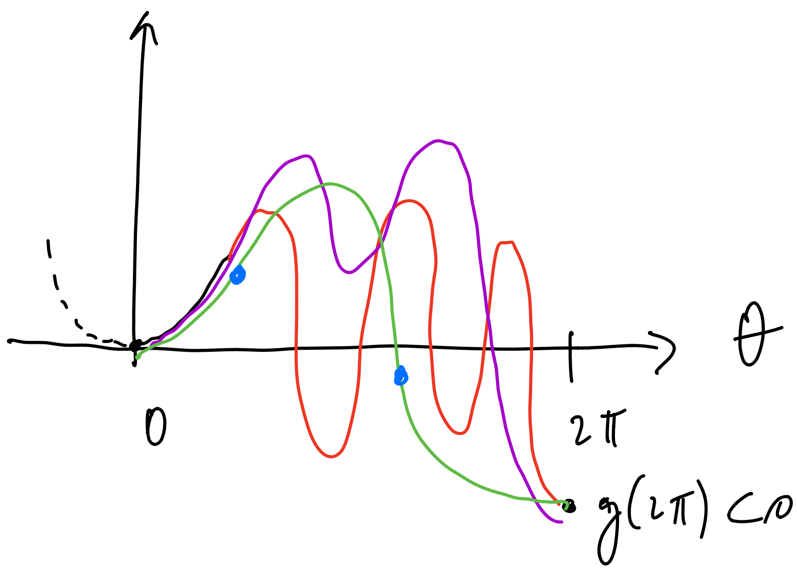
<u>Lema</u>: Funkcija g ima na $(0, 2\pi)$ <u>natanko eno ničlo</u>.

<u>Dokaz</u>:

Vemo: $g(0) = 0$ in $g(2\pi) < 0$.

$g'(0) = 0$ in $g''(0) = 1 > 0$

$\Rightarrow$ g ima na $(0, 2\pi)$ <u>vsaj eno ničlo</u>.



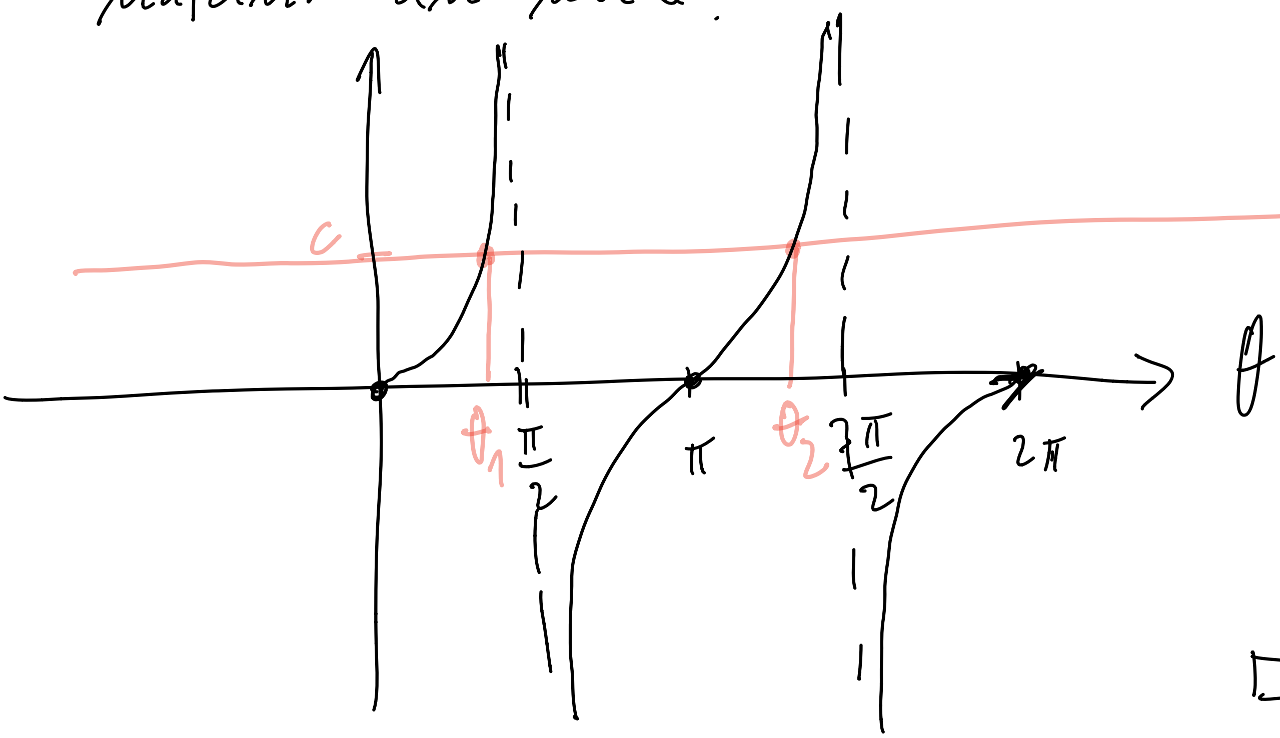
Dovolj je videti, da ima g'' natanko

dve ničli na $(0, 2\pi)$.

$$g''(\theta) = \cos\theta + \frac{B}{L}\sin\theta \overset{?}{=} 0$$

$$\operatorname{tg}\theta = \frac{\sin\theta}{\cos\theta} = -\frac{B}{L} \geq 0 \quad =: c \geq 0$$

Preveriti moramo, da ima ta enačba natanko dve ničli!



$\square$

Dokazali smo, da obstaja natanko en $\theta^* \in (0, 2\pi)$:

$$x(\theta) = \frac{k^2}{2}(\theta - \sin\theta),$$

$$y(\theta) = -\frac{k^2}{2}(1 - \cos\theta), \quad \theta \in [0, \theta^*]$$

je parametrizacija brahistohrone.

Čas potovanja po brahistohroni

Vemo, da je

$$t = \int_0^{L} \sqrt{\frac{1 + y'(x)^2}{-2 g y(x)}}\, dx$$

$$x(\theta) = \frac{1}{2} k^2 (\theta - \sin\theta), \qquad y(\theta) = -\frac{1}{2} k^2 (1 - \cos\theta), \qquad \theta \in [0, \theta^*).$$

$$\Rightarrow t = \int_0^{\theta^*} \sqrt{\frac{1 + \left(\frac{dy}{dx}(\theta)\right)^2}{-2 g y(\theta)}}\, dx(\theta)$$

$$= \int_0^{\theta^*} \sqrt{\frac{1 + \left(\frac{\cancel{-\frac{1}{2}k^2} \sin\theta}{\cancel{\frac{1}{2}k^2}(1-\cos\theta)}\right)^2}{-\cancel{2} g\left(-\frac{1}{\cancel{2}} k^2 (1-\cos\theta)\right)}}\, \frac{1}{2} k^2 (1-\cos\theta)\, d\theta$$

$$= \int_0^{\theta^*} \sqrt{\frac{1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}}{g k^2 (1-\cos\theta)}} \cdot \frac{1}{2} k^2 (1-\cos\theta)\, d\theta$$

$$= \int_0^{\theta^*} \sqrt{\frac{1-2\cos\theta+\underbrace{\cos^2\theta+\sin^2\theta}_{1}}{gk^2(1-\cos\theta)^3}}\,\frac{1}{2}k^2(1-\cos\theta)\,d\theta$$

$$= \int_0^{\theta^*} \sqrt{\frac{2\,\cancel{(1-\cos\theta)}}{gk^2(1-\cos\theta)^{\cancel{3}\,2}}}\,\frac{1}{2}k^2(1-\cos\theta)\,d\theta$$

$$\frac{x}{\sqrt{x^2}} = \operatorname{sign}(x)$$

$$= \int_0^{\theta^*} \sqrt{\frac{2}{gk^2}}\,\frac{1}{2}k^2\,\frac{\cancel{1-\cos\theta}}{\cancel{1-\cos\theta}}\,d\theta$$

$$= \frac{k}{\sqrt{2g}}\,\theta\Big|_0^{\theta^*} = \frac{k}{\sqrt{2g}}\,\theta^*$$