$$f(x) = \ln(x^2-1) + \frac{1}{x^2-1}$$

$D_f: (-\infty, -1) \cup (1, \infty)$

$f$ je soda $\left(f(-x) = f(x)\right) \Rightarrow$ osredotočimo se na $x \in (1, \infty)$

$$f'(x) = \frac{2x}{x^2-1} - \frac{2x}{(x^2-1)^2} =$$

$$= 2x\,\frac{x^2-1-1}{(x^2-1)^2} = \overbrace{\overset{\text{liha}}{2x}\,\overbrace{\frac{x^2-2}{(x^2-1)^2}}^{\text{soda}}}^{\text{liha}}$$

$x = \sqrt{2}$ je stac. točka $\Rightarrow$ $x = -\sqrt{2}$ je stac. točka

$f'(x) < 0$ za $x \in (1, \sqrt{2})$ $\Rightarrow$ $f$ padajoča na $(1, \sqrt{2})$

$f'(x) > 0$ za $x > \sqrt{2}$ $\Rightarrow$ $f$ naraščajoča na $(\sqrt{2}, \infty)$

$$f(\sqrt{2}) = 1$$

Sklepamo: $f$ nima ničel

$$f''(x) = \left[2x\,\frac{x^2-2}{(x^2-1)^2}\right]' =$$

$$= 2\,\frac{x^2-2}{(x^2-1)^2} + 2x\,\frac{2x(x^2-1)^{\cancel{2}} - (x^2-2)2\cancel{(x^2-1)}2x}{(x^2-1)^{\cancel{4}\,3}}$$

$$= 2 \frac{(x^2-2)(x^2-1) + x \cdot 2x (x^2-1) - (x^2-2) 4x^2}{(x^2-1)^3}$$

$$= 2 \frac{x^4 - 3x^2 + 2 + 2x^4 - 2x^2 - 4x^4 + 8x^2}{(x^2-1)^3}$$

$$= 2 \frac{-x^4 + 3x^2 + 2}{(x^2-1)^3}$$ ... soda

$t := x^2$

$t^2 - 3t - 2$

$$t_{1,2} = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2} \qquad \sqrt{17} \approx 4$$

$$t_1 = \frac{3+\sqrt{17}}{2} \Rightarrow x_1 = \sqrt{\frac{3+\sqrt{17}}{2}}$$ ... ničla drugega odvoda

$$\doteq \sqrt{\frac{7}{2}}$$

"med $\sqrt{3}$ in 2"

$\lim\limits_{x \to \infty} (-x^4 + 3x^2 + 2) = -\infty$, $f''$ ne zamenja predznaka do prevoja ($f''$ je definiran na

$\sqrt{3+\sqrt{17}}$

$(1, \infty)\Big| \Rightarrow f''(x) < 0$ za $x > \sqrt{\frac{\;}{2}}$

$f''(x) > 0$ za $1 < x < \sqrt{\frac{3+\sqrt{17}}{2}}$

Obnašanje $f$ na robovih def. območja:

$$\lim_{x \downarrow 1} f(x) = \lim_{x \downarrow 1} \Big[\underbrace{\ln(x^2-1)}_{\to -\infty} + \underbrace{\frac{1}{x^2-1}}_{\to \infty}\Big] =$$

$$= \lim_{x \downarrow 1} \underbrace{\frac{1}{x^2-1}}_{\to \infty} \Big[\underbrace{\ln(x^2-1) \cdot (x^2-1)}_{\to 0\,?} + 1\Big] = \infty$$

$=$

$t := x^2 - 1$

$$\lim_{t \downarrow 0} \ln t \cdot t = \lim_{t \downarrow 0} \frac{\ln t}{\frac{1}{t}} \overset{L'H}{=} \lim_{t \downarrow 0} \underbrace{\frac{\frac{1}{t}}{-\frac{1}{t^2}}}_{-t} = 0$$

$$\lim_{x \downarrow 1} e^{f(x)} = \lim_{x \downarrow 1} e^{\ln(x^2-1) + \frac{1}{x^2-1}} =$$

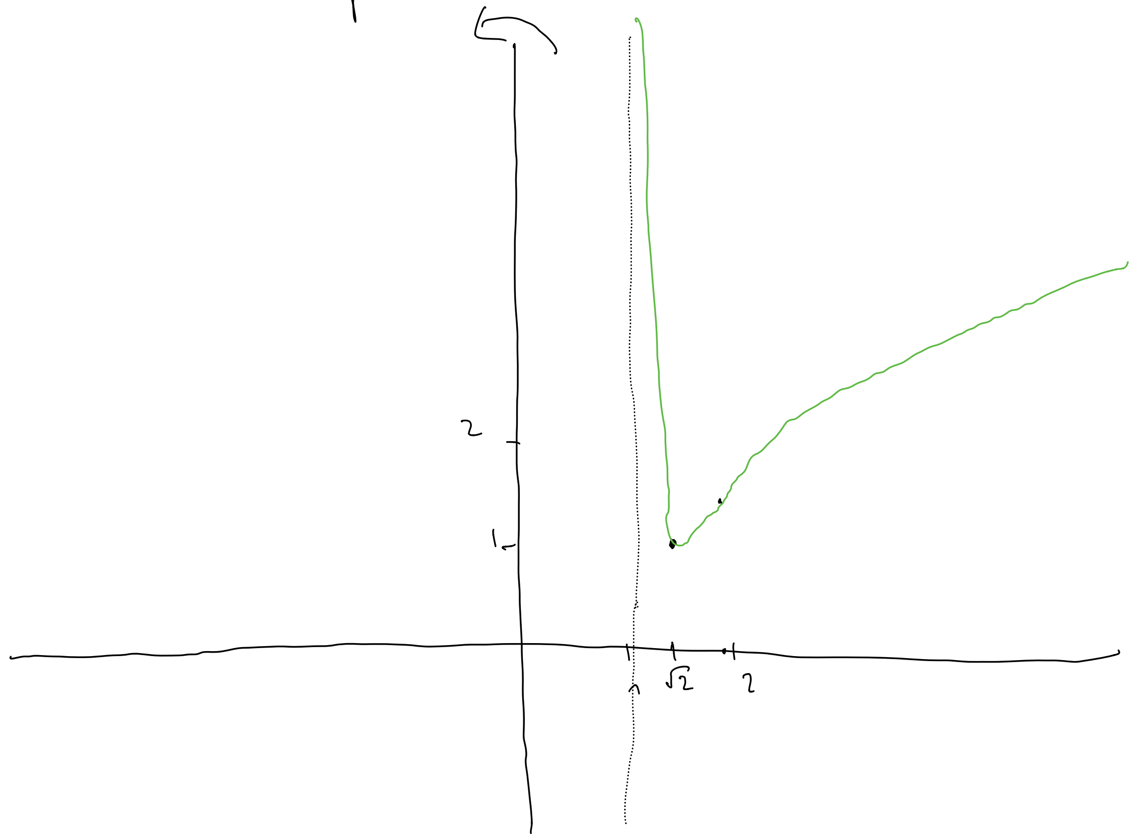
$$= \lim_{x \downarrow 1} (x^2-1) \cdot e^{\frac{1}{x^2-1}} =$$

$$\downarrow \qquad\qquad\qquad \infty$$
$$0$$

$$= \lim_{x \downarrow 1} \frac{e^{\frac{1}{x^2-1}}}{\frac{1}{x^2-1}} \overset{L'H}{=} \dots = \infty$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \underbrace{\ln(x^2-1)}_{\infty} + \underbrace{\frac{1}{x^2-1}}_{0} = \infty$$

prezrcalimo




$$x_1 := \sqrt{\frac{\sqrt{17}+3}{2}} \approx 1{,}9 \qquad f(x_1) = \ln\left(\frac{3+\sqrt{17}}{2} - 1\right)$$

$$+ \frac{1}{\frac{\sqrt{17}+3}{\dots}}$$

$\frac{\sqrt{17} \cdots}{2} - 1$

$\approx 1{,}4$

$f(x) = (2x^2 - 17)\sqrt[3]{x^2 - 1}$

$D_f : \mathbb{R}$

$f$ je soda $\Rightarrow$ osredotočimo se na $[0, \infty)$

Ničle: $x_1 = \sqrt{\frac{17}{2}}$ , $x_2 = 1$

$\sqrt{8}$

$f(0) = 17$

$f'(x) = 4x\sqrt[3]{x^2-1} + (2x^2 - 17)\frac{1}{3}\frac{2x}{(x^2-1)^{2/3}} =$

$\cdots\cdots = 2x\frac{8x^2 - 23}{3(x^2-1)^{2/3}} = \frac{2x(8x^2-23)}{3(x^2-1)^{2/3}}$

$f$ ni odvedljiva v $x = 1$ , $f(1) = 0$

Stac. točke $f$ : $x_1 = 0$

$x_2 = \sqrt{\frac{23}{8}} \approx \sqrt{\frac{24}{8}} = \sqrt{3}$

$f'$ lahko zamenja predznak v: - stac. točkah

- točka, kjer $f$ ni odvedljiva

V $x_2$ $f'$ zamenja predznak, v $x=1$ pa ne (imenovalec je vedno nenegativen)

$f'(x) > 0$ za $x > x_2$ $\Rightarrow$ $f$ naraščajoča na $(x_2, \infty)$

$f'(x) < 0$ za $x \in (0, x_2)$ $\Rightarrow$ $f$ pada na $(0, x_2)$

$$f(0) = 17, \quad f\left(\sqrt{\frac{23}{8}}\right) = \left(2\cdot\frac{23}{8} - 17\right)\sqrt[3]{\frac{23}{8} - 1} =$$

$$= \left(\frac{23}{4} - 17\right)\sqrt[3]{\frac{23-8}{8}} =$$

$$= \left(\frac{23 - 4\cdot 17}{4}\right)\sqrt[3]{\frac{15}{8}} =$$

med 2 in 3

$$= \frac{23-68}{4}\,\frac{\sqrt[3]{15}}{2} = \frac{-45\sqrt[3]{15}}{8}$$

$$f''(x) = \left[2x\,\frac{8x^2-23}{3(x^2-1)^{2/3}}\right]' = \ldots =$$

$$= \frac{80x^4 - 98x^2 + 138}{9(x^2-1)^{5/3}}$$

Zgoraj imamo kvadratno enačbo, ki pa nima realnih rešitev.

( diskr. $\quad 98^2 - 4 \cdot 80 \cdot 138 < 0$ )

$f''$ nima prevojev, a lahko zamenja predznak

v $x = 1$

$f''$ spremeni predznak v $x = 1$

$f''(x) > 0 \quad$ za $\quad x > 1$

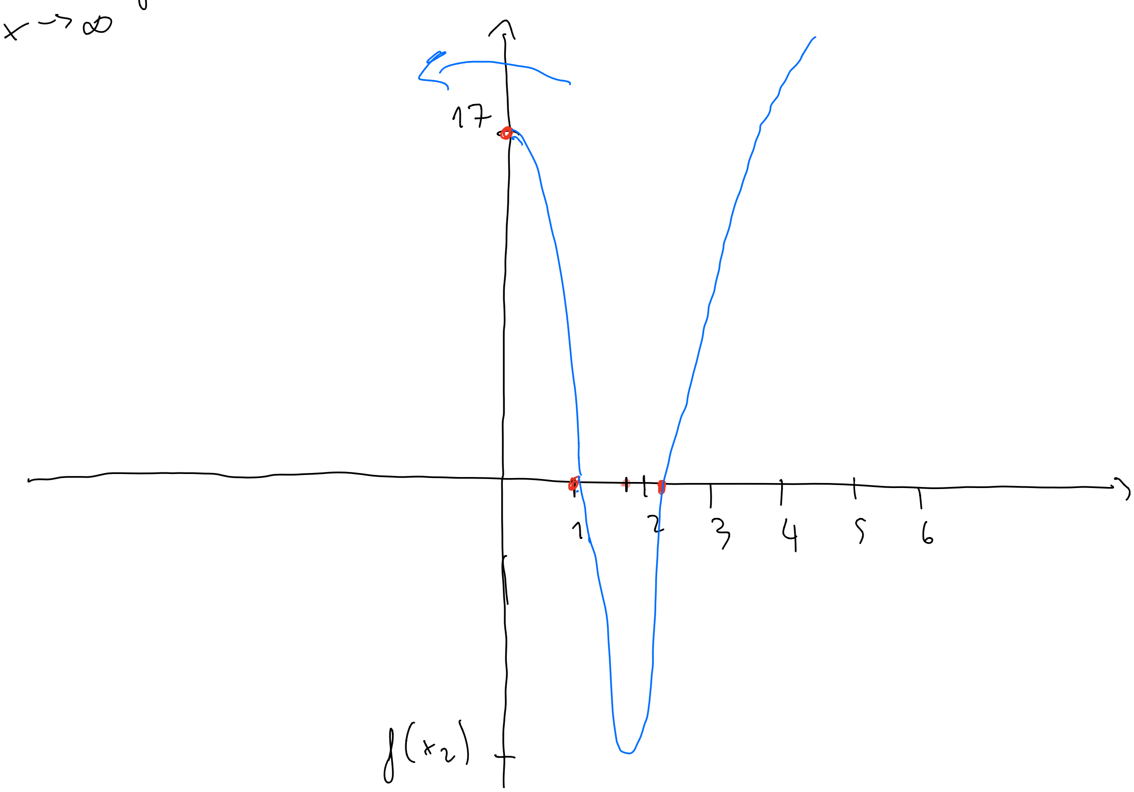
$f''(x) < 0 \quad$ za $\quad x \in (0, 1)$

$f'(x) \xrightarrow{x \to \infty} \infty$

$\Rightarrow$ $f$ nima

$\nearrow$ poševne asimptote

$$\lim_{x \to \infty} f(x) = (2x^2 - 17)\sqrt[3]{x^2 - 1} = \infty$$



Torek ob 16h ... G.U.

$f(x) = \frac{1}{x^2+2x}$ ; $f^{(n)}(x) = ?$

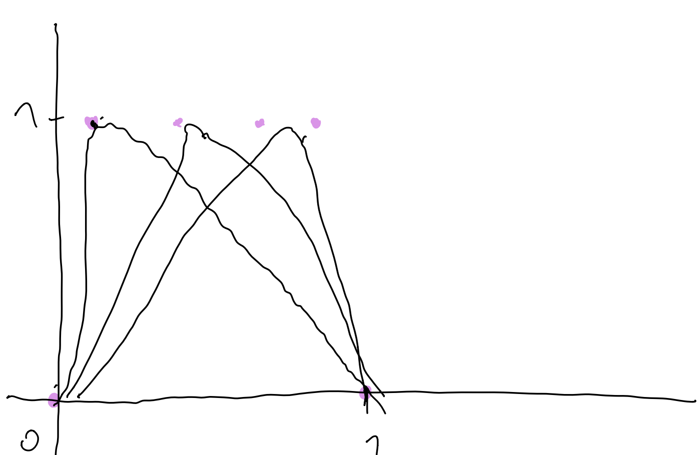
Ideja: ugane se vzorec, ki se ga dokaže z indukcijo.

$f: [0,1] \longrightarrow \mathbb{R}$, zv., zv. odvedljiva na $(0,1)$,

$f(0) = f(1) = 0$

$f(x_0) = 1$ za $x_0 \in (0,1)$

Dokaži, da obstaja $x_1 \in (0,1)$, da velja $|f'(x_1)| \geq 2$.



$x_0 \in (0,1)$, $f(x_0) = 1$

$x_0 \overset{*}{\leq} \frac{1}{2}$, $x_1 \in (0, \frac{1}{2})$

---

Lagrange: $\frac{f(b)-f(a)}{b-a} = f'(c)$, $c \in (a,b)$

$a = 0$, $b = x_0$

$f(a) = 0$, $f(b) = 1$

po L. izreku obstaja tak c

$$\frac{f(b)-f(a)}{b-a} = \frac{1}{x_0} = \frac{1}{x_0} = f'(c),$$

$c \in (0, x_0)$

$$\frac{1}{x_0} \overset{*}{\geq} 2 \Rightarrow f'(x_1) \geq 2$$

za neki $x_1 \in (0, x_0)$ ✓

$x_0 \geq \frac{1}{2}$

Podobno: $a = x_0$; $b = 1$

$$\frac{-1}{1-x_0} = \frac{1}{x_0 - 1} \leq \frac{1}{\frac{1}{2} - 1} = \frac{2}{1-2} = -2$$

$\Downarrow$ $x_0 \geq \frac{1}{2}$

$$\frac{-1}{1-x_0} = \frac{f(b)-f(a)}{b-a} = f'(x_1)$$

za nek

$x_1 \in (x_0, 1)$

$\Rightarrow f'(x_1) \leq -2$

$\Rightarrow |f'(x_1)| \geq 2$ ✓

Nalogo je mogoče rešiti, če dokažujemo, da je $|f'(x_1)| > 2$ za nek $x_1 \in (0,1)$.

# TAYLORJEVA VRSTA

Ideja ... funkcijo $f$ v ~~okoli~~ okolici neke točke aproksimiramo s polinomom

$f$ ... $n$-krat odvedljiva v okolici t. č. $a \in D_f$:

$$T_n(x) := f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

---

$R_n(x) = f(x) - T_n(x)$ $\xi$ med $x$ in $a$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

$\hookrightarrow$ $f$ $(n+1)$-krat zv. odv.

$R_n(x) \xrightarrow{n \to \infty} 0$, potem pravimo, da je $f$ analitična in velja

$$f(x) = T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$T_3(x)$, kjer je $T_3$

$p(x) = 2x^3 - 5x^2 + x + 7$, razvit okoli točke $a = 1$.

$$T_3(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$$

$a_0, \dots, a_3$ ?

$$a_0 = p(1) = 5$$

$$a_1 = \frac{p'(1)}{1!} = -3$$

$$a_2 = \frac{p''(1)}{2!} = \frac{2}{2!} = 1$$

$$a_3 = \frac{p'''(1)}{3!} = \frac{12}{6} = 2$$

$$p'(x) = 6x^2 - 10x + 1$$
$$p''(x) = 12x - 10$$
$$p'''(x) = 12$$

Izračunaj $e$ z napako, manjšo od $10^{-5}$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^1 = \underset{1}{f(0)} + \frac{\overset{1}{f'(0)}}{1!}\cdot(1-0) + \dots + \frac{\overset{1}{f^{(n)}(0)}}{n!} + \qquad e^0 = 1$$

$$+ \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot 1 \qquad f^{(n+1)}(\xi) = e^{\xi}$$

$\xi$ med 0 in 1

$$= \underbrace{1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}} + \underbrace{\frac{e^{\xi}}{(n+1)!}}_{\leq 10^{-5}}$$

$$\frac{e^{\xi}}{(n+1)!} \leq 10^{-5} \longrightarrow 1 \leq e^{\xi} \leq e \leq 3$$

$$\frac{e^{\xi}}{(n+1)!} \leq \frac{3}{(n+1)!} \leq 10^{-5} = \frac{1}{10^5}$$

$n = 6$ ?

$$\frac{3}{7!} > 10^{-5}$$

$n = 8$

$$\frac{3}{9!} < 10^{-5} \quad \checkmark$$

$$\overbrace{e - \underbrace{\left(1 + \frac{1}{1!} + \dots + \frac{1}{8!}\right)}} < 10^{-5}$$

Izračunajte $\cos\left(\frac{1}{2}\right)$ na pet dec. mest natančno.