Razvij f v Taylorjevo vrsto okoli točke a.

$f(x) = \frac{1}{x}$, $a = 1$

Cilj: za vse $n \geq 0$ izračunati $a_n \in \mathbb{R}$, da velja $f(x) = \sum_{n \geq 0} a_n (x-a)^n$ za vse x v neki okolici a.

$x - 1 := t$, $x = 1 + t$

$$\frac{1}{x} = \frac{1}{1+t} = \frac{1}{1-(-t)} = \sum_{n \geq 0} (-t)^n = \sum_{n \geq 0} (-1)^n (x-a)^n$$

$R = 1$

$f(x) = \frac{1}{x^2 - 5x + 6}$, $a = 1$

$$f(x) = \frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \quad ; \quad A, B \in \mathbb{R}$$

razcep na parcialne ulomke

$$\frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3) + B(x-2)}{(x-2)(x-3)} = \frac{(A+B)x + (-3A-2B)}{(x-2)(x-3)}$$

$$= \frac{1}{(x-2)(x-3)}$$

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$$\Rightarrow \begin{array}{l} A + B = 0 \Rightarrow B = -A \\ -3A - 2B = 1 \end{array}$$

$-A = 1 \Rightarrow A = -1, \; B = 1$

$$f(x) = -\frac{1}{x-2} + \frac{1}{x-3} =$$

$\hookrightarrow$ $t = x-1$

$x = 1+t$

$$= -\frac{1}{t-1} + \frac{1}{t-2} =$$

$$= \frac{1}{1-t} - \frac{1}{2\left(1-\frac{t}{2}\right)} =$$

za $\left|\frac{t}{2}\right| < 1$

za $|t| < 1$

$$= \sum_{n\geq 0} t^n - \frac{1}{2}\sum_{n\geq 0}\left(\frac{t}{2}\right)^n =$$

$$= \sum_{n\geq 0}(x-1)^n - \frac{1}{2}\sum_{n\geq 0}\frac{1}{2^n}(x-1)^n =$$

$$= \sum_{n\geq 0}\left(1-\frac{1}{2^{n+1}}\right)(x-1)^n \; ;$$

$$\frac{\underbrace{2^{n+1}-1}}{2^{n+1}}$$

R?

$|x-1| < 1$

$R = 1$

$f(x) = \sin x, \; a = \frac{\pi}{4}$

Vemo: $\sin x = \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$ za $x \in \mathbb{R}$

$\hookrightarrow$ razvoj sin okoli $a=0$

π t

$t = x - \frac{\pi}{4}$, $x = t + \frac{\pi}{4}$

$$\sin x = \sin\left(t + \frac{\pi}{4}\right) =$$

$$= \sin t \cos\left(\frac{\pi}{4}\right) + \cos t \sin\left(\frac{\pi}{4}\right) =$$

$$= \frac{\sqrt{2}}{2}\left(\sin t + \cos t\right) =$$

$$= \frac{\sqrt{2}}{2}\left(\sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} t^{2n+1} + \sum_{n \geq 0} \frac{(-1)^n}{(2n)!} t^{2n}\right) =$$

$$= \frac{\sqrt{2}}{2} \sum_{n \geq 0} \frac{(-1)^{f(n)}}{n!} t^n$$

$$f(n) = \begin{cases} 0 \; ; & n \equiv 0, 1 \ (4) \\ 1 \; ; & n \equiv 2, 3 \ (4) \end{cases}$$

$R = \infty$

$f(x) = \arctan x$, $a = 0$ $(t = x)$

$$f'(x) = \frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} =$$

$$= \sum_{n=0}^{\infty} (-1) x^{2n}$$

$\hookrightarrow$ $|x^2| < 1$

$|x| < 1$

Členoma integriramo, dobimo

$$f(x) = \sum^{\infty} (-1) \frac{x^{2n+1}}{2n+1} + C$$

vstavimo $x = 0$ $\Longrightarrow$ $C = 0$

$n=0$

$R = 1$

$$f(x) = \ln\left(\frac{\sqrt{x^2+1}}{2x^2+1}\right), \quad a = 0$$

$$\ln(1+x) = \sum_{n\geq 1} (-1)^{n+1} \frac{x^n}{n}$$

$|x| < 1$

$2x^2 + 1 = (\sqrt{2}x)^2$

$|x| < 1$

$|\sqrt{2}x| < 1$

$$\ln\left(\frac{\sqrt{x^2+1}}{2x^2+1}\right) = \frac{1}{2}\ln(1+x^2) - \ln(2x^2+1) =$$

$$= \frac{1}{2}\sum_{n\geq 1} (-1)^{n+1}\frac{x^{2n}}{n} - \sum_{n\geq 0} (-1)^{n+1}(\sqrt{2})^n \frac{x^{2n}}{n} =$$

$$= \sum_{n\geq 1} (-1)^{n+1} \frac{\left(\frac{1}{2} - (\sqrt{2})^n\right)}{n} x^{2n} =$$

$$= \sum_{n\geq 1} (-1)^{n+1} \frac{1 - 2(\sqrt{2})^n}{2n} x^{2n}$$

$R = \frac{\sqrt{2}}{2}$

$$f(x) = \sin^2 x \; ; \quad a = 0$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x) =$$

$$= \frac{1}{2}\left(1 - \sum_{n\geq 0} (-1)^n \frac{x^{2n}}{(2n)!}\right) =$$

$$= \sum_{n\geq 1} (-1)^{n+1} \frac{x^{2n}}{2(2n)!} \qquad R = \infty$$

$$f(x) = \ln(1 + x + x^2) \,; \quad a = 0$$

$$(1 + x + x^2)(1 - x) = 1 - x^3$$

$$f(x) = \ln\left(\frac{1 - x^3}{1 - x}\right) = \ln(\underbrace{1 - x^3}) - \ln(1 - x) =$$

$$= -\sum_{n=1}^{\infty} \frac{x^{3n}}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} a_n x^n$$

$|x| < 1$

$$\boxed{\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad (\text{dokaz enak kot za } \ln(1+x))}$$

$|x| < 1$

$$a_n = \begin{cases} \frac{1}{n} & ; \quad n \not\equiv 0 \ (3) \\ -\frac{2}{n} & ; \quad n \equiv 0 \ (3) \end{cases}$$

$$\hookrightarrow \frac{1}{n} - \frac{1}{\frac{n}{3}}$$

$$R = 1$$