Razvij f v Taylorjevo vrsto v okolici točke a.

$f(x) = \frac{1}{x}$, $a = 1$

Hočemo $a_n \in \mathbb{R}$, za vse $n \geq 0$, da je

$$f(x) = \sum_{n \geq 0}^{\infty} a_n (x-a)^n \quad \text{za vse } x \text{ v neki}$$

okolici a.

$$t := x - 1 \Rightarrow x = 1 + t$$

$$\frac{1}{x} = \frac{1}{1+t} = \sum_{n \geq 0} a_n t^n$$

$$\left[\frac{1}{1-t} = \sum_{n \geq 0} t^n \quad \text{(Razvij v geometrijsko vrsto)} \right]$$

$|t| < 1$

$$\underbrace{\sum_{n=0}^{N} t^n}_{(1+t+\dots+t^N)} \cdot (1-t) = (1 - t^{N+1}) \xrightarrow{N \to \infty} 1$$

$|t| < 1$

$$\frac{1}{1+t} = \frac{1}{1-(-t)} = \sum_{n \geq 0}^{\infty} (-t)^n = \boxed{\sum_{n \geq 0}^{\infty} (-1)^n (x-1)^n}$$

$|t| < 1$

$|x - 1| < 1$

$R = 1$

$f(x) = \frac{1}{x^2 - 5x + 6}$; $a = 1$

$$\frac{1}{x^2-5x+6} = \underbrace{\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}}, \quad A, B \in \mathbb{R}$$

Razcep na parcialne ulomke

$$\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} =$$

$$= \frac{A(x-2) + B(x-3)}{(x-3)(x-2)}$$

$x:\quad 0 = A + B \Rightarrow B = -A$

$1:\quad 1 = -2A - 3B \Rightarrow -2A + 3A = A$

$\Rightarrow A = 1, \quad B = -1$

$$f(x) = \frac{1}{x-3} - \frac{1}{x-2} =$$

$t := x - 1$

$$= \frac{1}{t-2} - \frac{1}{t-1} =$$

$$= -\frac{1}{2\left(1-\frac{t}{2}\right)} + \frac{1}{1-t} = \quad \cdot \left|\frac{t}{2}\right| < 1 \quad \cdot |t| < 1$$

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

$\hookrightarrow |t| < 1$

$a\ \frac{1}{1-t'}$

$$= \sum_{n\geq 0}^{\infty} t^n - \frac{1}{2}\sum_{n\geq 0}^{\infty}\left(\frac{t}{2}\right)^n =$$

$$= \sum_{n\geq 0}^{\infty}\left(1 - \frac{1}{2}\cdot\left(\frac{1}{2}\right)^n\right)(x-1)^n$$

$$= \sum_{n\geq 0}^{\infty} \frac{2^{n+1}-1}{2^{n+1}}(x-1)^n$$

$f(x) = \sin x \;;\quad a = \frac{\pi}{4}$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$(n)\ (\pi)$

Neposredno: izr. $\sin^{(n)}\left(\frac{\pi}{4}\right)$, $n = 0, \dots, 3$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$t = x - \frac{\pi}{4}$$

$$\sin x = \sin\left(t + \frac{\pi}{4}\right) = \sin t \cos\frac{\pi}{4} + \cos t \sin\frac{\pi}{4} =$$

$$= \frac{\sqrt{2}}{2}\left(\sin t + \cos t\right) \overset{t \in \mathbb{R}}{=}$$

$$= \frac{\sqrt{2}}{2}\left(1 + t - \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \dots\right)$$

$$= \sum_{n \geq 0}^{\infty} a_n \left(x - \frac{\pi}{4}\right)^n$$

$$a_n = \begin{cases} +\dfrac{\sqrt{2}}{2 \cdot n!} & n = 0, 4, 8, \dots \\[2ex] \dfrac{\sqrt{2}}{2\, n!} & n = 1, 5, 9, \dots \\[2ex] -\dfrac{\sqrt{2}}{2\, n!} & \text{sicer} \\ & (n \equiv 2 \text{ ali } n \equiv 3 \text{ mod. } 4) \end{cases}$$

$$R = \infty$$

$$f(x) = \arctan x \; ; \quad a = 0$$

$$f'(x) = \underbrace{\frac{1}{1 + x^2}} = \underbrace{\frac{1}{1 - (-x^2)}} = \quad \underline{\begin{array}{l} |x^2| < 1 \\ |x| < 1 \end{array}}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$0 = \arctan 0 + C \implies C = 0$$

$$R = 1$$

$$f(x) = \ln\left(\frac{\sqrt{x^2+1}}{2x^2+1}\right) ; \quad a = 0$$

$$\ln\left(\frac{\sqrt{x^2+1}}{2x^2+1}\right) = \ln(x^2+1)^{\frac{1}{2}} - \ln(2x^2+1) =$$

$$= \frac{1}{2}\ln(x^2+1) - \ln(2x+1) =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n} - \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\sqrt{2}x)^{2n}}{n}$$

$$\ln(1+x) =$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

$$= *$$

$$|x| < 1$$

$$|x^2| < 1$$

$$|(\sqrt{2}x)^2| < 1$$

$$|x| < \frac{\sqrt{2}}{2}$$

$$\ln(1+2x^2) = \ln(1+(\sqrt{2}x)^2)$$

$$* = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\frac{1}{2}-(\sqrt{2})^n}{n}x^{2n} =$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1-2(\sqrt{2})^n}{2n}x^{2n}$$

$$R = \frac{\sqrt{2}}{2}$$

$$f(x) = \sin^2 x \quad ; \qquad a = 0$$

$$\sin x \cdot \sin x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left(x - \frac{x^3}{3!} + \dots\right)$$

$$\cos 2x = \cos^2 x - \sin^2 x =$$

$$= 1 - 2\sin^2 x$$

$$\cos^2 x + \sin^2 x = 1$$

$$\frac{1}{2}\left(\cos 2x - 1\right) = -\sin^2 x$$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos 2x\right) =$$

$$= \frac{1}{2}\left(1 - \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!}(-1)^n\right) =$$

$$= \frac{1}{2}\left(- \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}(-1)^n x^{2n}\right) =$$

$$= \underline{\underline{\sum_{n=1}^{\infty} \frac{2^{2n-1}}{(2n)!}(-1)^{n+1} x^{2n}}}$$

$$f(x) = \ln(1 + x + x^2) \quad ; \quad a = 0$$

$$1 + x + x^2 = \frac{1 - x^3}{1 - x}$$

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

$|x| < 1$

$$f(x) = \ln(1 - x^3) - \ln(1 - x) =$$

$$= -\sum_{n=1}^{\infty} \frac{x^{3n}}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n} = \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$x \in \mathbb{R}$

S pomočjo zgornjega razvoja v Taylorjevo vrsto izr. Taylorjevi vrsti za sinh in cosh, razvite okoli $x = 0$.

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad ; \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

sinh ... lihi del e

cosh ... sodi del e

$$\sinh = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$\cosh = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

f ... trikrat odvedljiva funkcija

- $f(\frac{\pi}{2}) = \pi$
- $[\sin x + \sin(f(x)) = 1]^{*}$ za vse $x \in \mathbb{R}$

$T_3(f)$, razvit okoli $\frac{\pi}{2}$

$$T_3(f) = f(\tfrac{\pi}{2}) + \frac{f'(\frac{\pi}{2})}{1}(x-\tfrac{\pi}{2}) + \dots + \frac{f'''(\frac{\pi}{2})}{3!}(x-\tfrac{\pi}{2})^3$$

f':

$* \Rightarrow \sin' x + \sin'(f(x))\, f'(x) = 0$

$$\cos x + \cos(f(x))\, f'(x) = 0$$

$f'(\frac{\pi}{2})$: $0 - f'(\frac{\pi}{2}) = 0$

f'':

$* \Rightarrow$

$$\sin x - \sin(f(x))\, f'(x)\, f'(x) + \cos(f(x))\, f''(x)$$

$= 0$

Vstavimo $x = \frac{\pi}{2}$, dobimo:

$$1 - f''\left(\frac{\pi}{2}\right) = 0 \Rightarrow f''\left(\frac{\pi}{2}\right) = 1$$

f''':

$$\underset{x=\frac{\pi}{2}: \overset{\shortparallel}{0}}{\cos x} - \cos(f(x))\, \underset{\overset{\shortparallel}{0}}{f'(x)^3} - \underset{\overset{\shortparallel}{0}}{\sin(f(x))}\left(f'(x)^2\right)' +$$

$$- \underset{\overset{\shortparallel}{0}}{\sin}(f(x))\, f'(x) + \underset{\overset{\shortparallel}{-1}}{\cos(f(x))}\, f'''(x) = 0$$

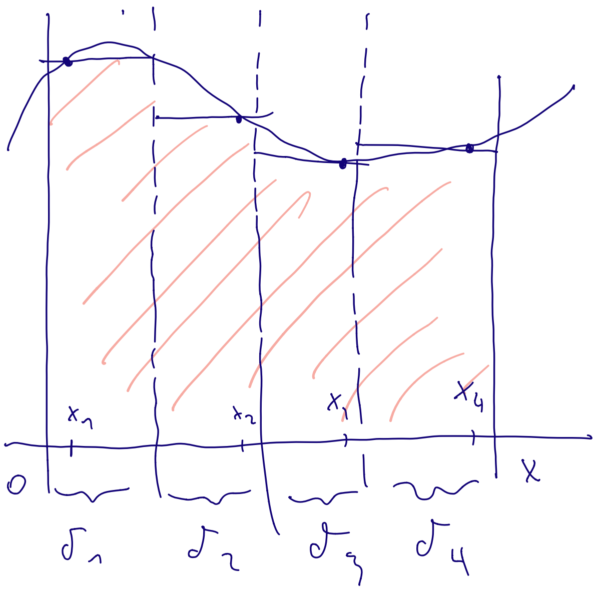
vst. $x = \frac{\pi}{2}$:

$$\Rightarrow f'''\left(\frac{\pi}{2}\right) = 0$$

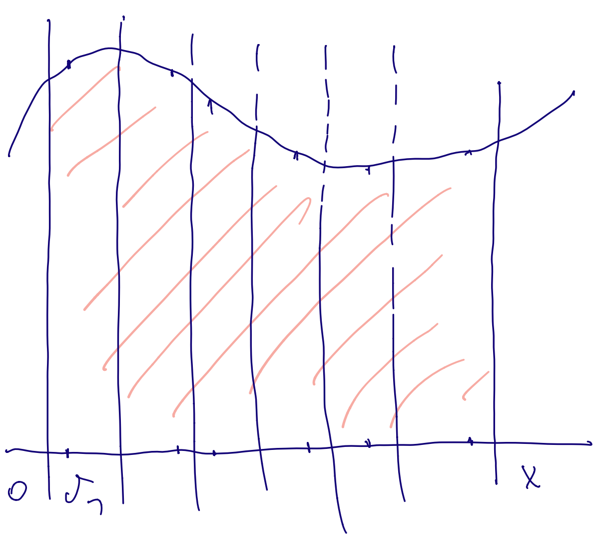
Dobimo 3. Taylorjev polinom

Riemannov integral

DOLOČENI INTEGRAL



$$S_1 = \sigma_1 \cdot f(x_1) + \ldots + \sigma_4 f(x_4)$$



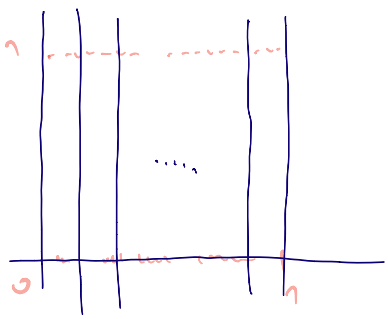
$$S_2 = \sigma_1 \cdot f(x_1) + \ldots + \sigma_6 f(x_6)$$

. . . .

$\delta_k := \max\{\delta_i\}$ → v k-ti iteraciji

$\delta \longrightarrow 0$ .... $\delta_k \xrightarrow{k \to \infty}$?

$$\cdot\ f(x) = \begin{cases} 1; & x \in \mathbb{Q} \\ 0; & x \notin \mathbb{Q} \end{cases}$$



• f ni R. integrabilna, ker so rac. in irac. št. gosta (sp. in zg. Darbouxova vsota sta različni)

$$\cdot\ f(x) = \begin{cases} 1; & x = \frac{1}{n};\ n \in \mathbb{N} \\ 0; & \text{sicer} \end{cases}$$

A je f Riemannovo integrabilna? → na $[0, 1]$

Izkaže se, da je. Velja celo $\int_0^1 f(x)\,dx = 0$

Dokaz: za vsak $\varepsilon > 0$ bomo našli $\delta > 0$, da bo za vsako delitev intervala, kjer bo max. dolžina manjših intervalov, manjša od δ, $\sum_{i=1}^{N} \delta_i f(x_i) < \varepsilon$

↳ za vsako izbiro točk $x_i \in [y_i, y_{i+1}]$

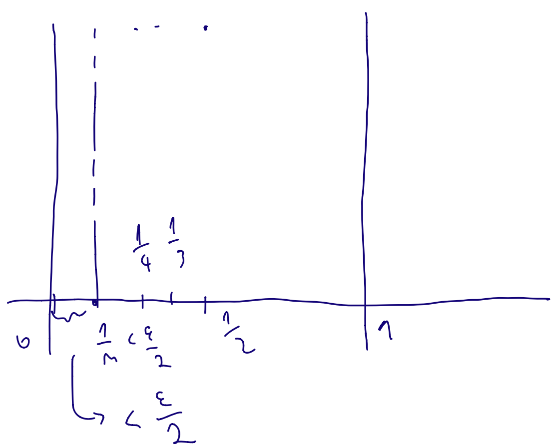
↓

krajišča manjših intervalov

$\varepsilon > 0$

Obstaja tak n, da velja:

$\frac{1}{n} < \frac{\varepsilon}{2}$



Particija intervala $[0, 1]$:

$\{x_0, x_1, x_2, \dots, x_k\}$

$x_0 = 0$

$x_1 = \frac{1}{n+1}$

x_2

x

'' }

...

x_k

$$x_{i+1} - x_i < \frac{\varepsilon}{4n}$$

Zgornje vsote: če je na intervalu $f(x) = 1$ za nek x iz intervala, vzamemo za testno točko v intervalu kar x

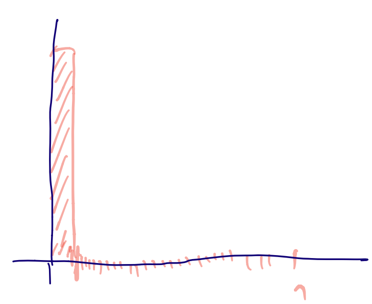
$$\delta_i < \frac{\varepsilon}{4n}$$

Zgornja vsota:

$$\frac{1}{n+1} + 2n\delta_i < \frac{\varepsilon}{2} + 2n \cdot \frac{\varepsilon}{4n} = \varepsilon$$

↳ na največ $2n$ intervalih bo maksimum f na tem intervalu enak 1

$x = \frac{1}{n}$ ravno na robovih, prištejemo vsako točko dvakrat.



Z Riemannovimi vsotami lahko rešujemo nekatere konkretne limite.

$k \geq 1$

$$\lim_{n \to \infty} \frac{1^k + 2^k + \dots + n^k}{n^{k+1}} = ? \qquad (k = 1, 2, 3, \dots)$$

Ta izraz zapišemo kot Riemannovo vsoto

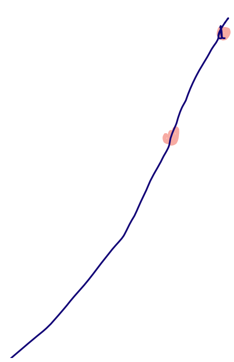
$$\sum_{i=0}^{n-1} (x_{i+1} - x_i) f(y_i) \quad ; \quad y_i \in [x_i, x_{i+1}]$$

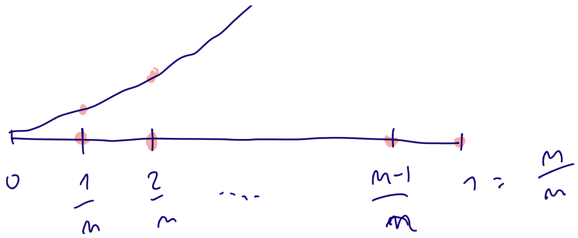
$$[x_0, x_1, x_2, \dots, x_n]$$

$$\frac{1^k + 2^k + \dots + n^k}{n^{k+1}} = \frac{1}{n} \frac{1^k + \dots + n^k}{n^k} =$$

$$= \sum_{i=1}^{n} \frac{1}{n} \cdot \underbrace{\left(\frac{i}{n}\right)^k}_{f\left(\frac{i}{n}\right)} \xrightarrow{n \to \infty} \int_0^1 x^k \, dx = \left. \frac{x^{k+1}}{k+1} \right|_0^1 =$$

$$= \frac{1}{k+1}$$





$$f : [0,1] \longrightarrow (0, \infty)$$

Poenostavi:

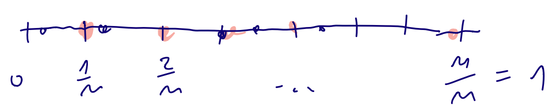
$$\lim_{n \to \infty} \sqrt[n]{f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \ldots f\left(\frac{n}{n}\right)} = *$$

$$\ln\left(\sqrt[n]{f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \ldots f\left(\frac{n}{n}\right)}\right) =$$

$$= \frac{1}{n} \ln\left(f\left(\frac{1}{n}\right) \cdot \ldots \cdot f\left(\frac{n}{n}\right)\right) =$$

$$= \sum_{k=1}^{n} \frac{1}{n} \underline{\ln\left(f\left(\frac{k}{n}\right)\right)}$$

$$\Big\downarrow \; n \longrightarrow \infty$$



$$\int_0^1 \ln(f(x))\, dx$$

$$\int^1 \ln(f(x))\, dx$$

$$\# = e$$

$$\lim_{n \to \infty} \underbrace{\left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n} \right)}_{2n \text{ členov vsote}}$$

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+2n} =$$

$$= \frac{1}{2n} \left(\frac{1}{\frac{1}{2} + \frac{1}{2n}} + \frac{1}{\frac{1}{2} + \frac{2}{2n}} + \frac{1}{\frac{1}{2} + \frac{3}{2n}} + \dots + \frac{1}{\frac{1}{2} + \frac{2n}{2n}} \right)$$

$$\frac{2n}{2n} = 1$$

$0 \quad \frac{1}{2n} \quad \frac{2}{2n} \quad 1 = \frac{2n}{2n}$

Kaj bodo delilne točke?

Kaj bodo testne točke?

Kaj bo f?

$$\xrightarrow{n \to \infty}$$

$$\int_0^1 \frac{1}{\frac{1}{2} + x} \, dx$$