1. kolokvij iz Matematike 1

11. december 2020

Veliko uspeha!

1. naloga (25 točk)

Dokaži, da za vsako naravno število n velja

$$\underbrace{2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2}_{S_n} = \frac{1}{2}\underbrace{n(6n^2 + 3n - 1)}_{a_n}.$$

INDUKCIJA

(1) $n = 1$: $3n - 1 = 2 \implies S_n = 2^2 = 4$

$$a_n = \frac{1}{2}\cdot 1\,(\underbrace{6\cdot 1^2 + 3\cdot 1 - 1}_{8}) = 4$$

(5)

(2) I.K.: $S_{n+1} = S_n + (\underbrace{3(n+1)-1}_{3n+2})^2 \overset{\text{I.P.}}{=}$ (20)

$$= \overbrace{\frac{1}{2}n(6n^2+3n-1)}^{3n^3 + \frac{3}{2}n^2 - \frac{1}{2}n} + 9n^2 + 12n + 4 =$$

$$= 3n^3 + \frac{21}{2}n^2 + \frac{23}{2}n + 4$$

$$a_{n+1} = 3(n^3 + 3n^2 + 3n + 1) + \frac{3}{2}(n^2 + 2n + 1) - \frac{1}{2}(n+1) =$$

$$= 3n^3 + \frac{21}{2}n^2 + \frac{23}{2}n + 4$$

2. naloga (25 točk)

Reši enačbo

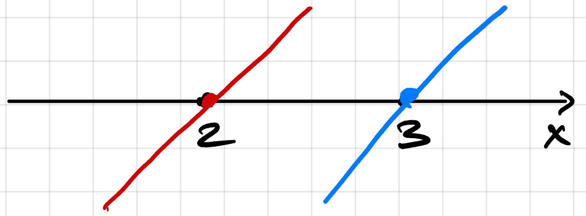
$$|\overbrace{2x^2 - 8}^{2(x^2-4)}| = |x^2 - x - 6|.$$

$$|2(x-2)(x+2)| = |(x+2)(x-3)|$$

$$|2x-4|\,|x+2| = |x+2|\,|x-3|$$ (5)

(1) $|x+2| = 0 \iff x = -2$ (5)

(2) $|x+2| \neq 0$: $2|x-2| = |x-3|$



(2.1) $x \leq 2$: "$-$ $-$" $\implies -2(x-2) = -(x-3)$ (5)

$$-2x + 4 = -x + 3 \iff 1 = x, \text{ ker } 1 \leq 2.$$

2.2 $x \in [2,3]$: "$+\,-$" $\Longrightarrow 2x-4 = -(x-3) \Longrightarrow$ 5

$\Longrightarrow 3x = 7 \Longrightarrow x = \frac{7}{3} \in [2,3]$

2.3 $x \geq 3$: $2(x-2) = x-3 \Longrightarrow \cancel{x=1} \ngeq 3$ 5

3. naloga (25 točk)

Določi vse kompleksne rešitve enačbe

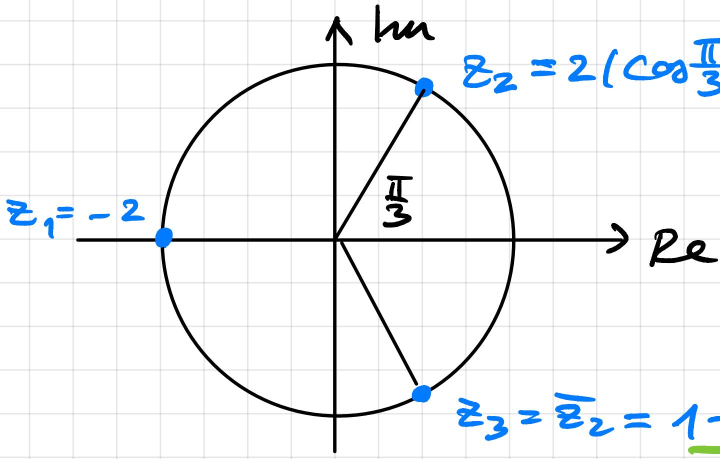
$$z^6 + 7z^3 = 8$$

in jih nariši v kompleksni ravnini.

$w = z^3$: $w^2 + 7w - 8 = (w+8)(w-1) = 0$ 5

1. $w = -8$: $z^3 = -8$, $z = \sqrt[3]{-8}$ 10

Ugenemo $z_1 = -2$, ostala korena pa tvorita enakostranični trikotnik na krožnici $|z| = 2$:



$z_2 = 2(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}) = 1 + i\sqrt{3}$

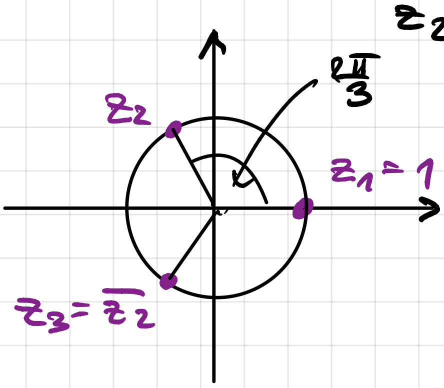
$z_3 = \overline{z_2} = 1 - i\sqrt{3}$

2. $w = 1$: $z^3 = 1$, $z = \sqrt[3]{1}$ 10

Vemo, da je $z_1 = 1$, ostala korena pa na krožnici $|z| = 1$ skupaj z z_1 določata enakostr. trikotnik:

$z_2 = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$



4. naloga (25 točk)

Dana je množica
$$A = \left\{ \frac{3n+2}{n+1} \; ; \; n \in \mathbb{N} \right\}.$$

Ugotovi, katere od količin $\min A$, $\inf A$, $\max A$ in $\sup A$ obstajajo ter jih določi. Vse odgovore natančno utemelji.

$A = \{ a_n = \frac{3n+2}{n+1} \; ; \; n \in \mathbb{N} \}$

$a_n = \frac{3n+3-1}{n+1} = 3 - \frac{1}{n+1}$ naraščajoče zaporedje,

zato je $a_1 = \frac{5}{2} = \min A = \inf A$ 10

Dokažimo, da je $3 = \sup A$:

① $a_n \leq 3$ za $\forall n$: $3 - \frac{1}{n+1} \leq 3$ ✓ 5

② $\beta < 3 \Rightarrow \exists a_n > \beta$:

$3 - \frac{1}{n+1} > \beta \Leftrightarrow \underset{\vee \atop 0}{3-\beta} > \frac{1}{n+1} \Leftrightarrow$

$\frac{1}{3-\beta} < n+1 \Leftrightarrow \underbrace{n > \frac{1}{3-\beta} - 1}$

Tak $n \in \mathbb{N}$ obstaja! ✓ 10

Ker je $\sup A = 3 \notin A$, $\max A$ ne obstaja. 2